Obliquely reflecting diffusions in curved, nonsmooth domains

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joint work with Thomas G. Kurtz

ROSCOFF
40 YEARS OF
REFLECTED BROWNIAN MOTION
AND RELATED TOPICS
APRIL 24>28
2023

Some examples from stochastic networks

W.N. Kang and R. J. Williams (2012): Diffusion approximation for an input-queued switch operating under a maximum weight matching policy, Stochastic Systems, 2, 277-321

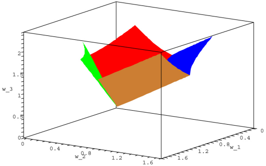


FIG 4. *A portion of the workload cone $\mathcal{W}_2$ is shown for a 2×2 input-queued switch with* $\alpha = 2$.

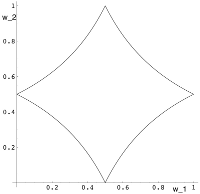


FIG 5. *A cross-section of the workload cone $\mathcal{W}_2$ depicted in Figure 4 taken at* $w_3 = 1$.

Some examples from stochastic networks

W.N. Kang, F.P. Kelly, N.H. Lee and R.J. Williams (2009): State space collapse and diffusion approximation for a network operating under a fair bandwidth sharing policy, The Annals of Applied Probability, 19, 1719-1780

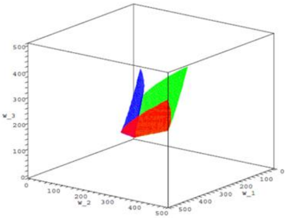


FIG. 8. *A portion of the workload cone $\mathcal{W}_{0.5}$ is shown for a linear network with three resources and four routes with $\alpha = 0.5$ and $\nu_i = \mu_i = \kappa_i = 1$ for all $i \in \mathbb{I}$.*

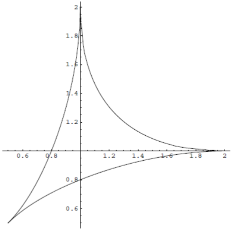
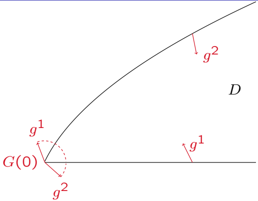


FIG. 9. *A cross-section of the workload cone $\mathcal{W}_{0.5}$ depicted in Figure 8 taken at $w_3 = 1$.*

Semimartingale reflecting diffusions



Stochastic Differential Equation with Reflection (SDER)

$$X(t) = X_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t \gamma(s)d\lambda(s),$$

$$X(s) \in \overline{D}, \quad \gamma(s) \in G(X(s)),\ G(x) \text{ a cone}, \quad |\gamma(s)| = 1,\ d\lambda - a.e.,$$

$$\lambda \ \text{ nondecreasing, continuous}, \quad d\lambda\big(\{s \le t :\ X(s) \in \partial D\}\big) = \lambda(t),$$

X is a solution if there exist W, γ, λ s.t. SDER is satisfied

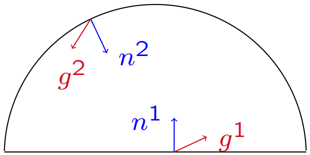
For non semimartingale reflecting diffusions: Ramanan (2006), Ramanan-Reiman (2008), Kang-Ramanan (2010), Lakner-Reed-Zwart (2017), etc.

For normal reflection: Tanaka (1979), Saisho (1987), Bass-Hsu (1990), De Blassie-Toby (1993), Z.-Q. Chen (1993), Bass-Burdzy (2006, 2008), etc.

Semimartingale obliquely reflecting diffusions: state of art

- Seminal works: Harrison and Reiman (1981), Varadhan and Williams (1984), Lions and Sznitman (1984), Williams (1985), Reiman and Williams (1988), Bernard and El Kharroubi (1991), Taylor and Williams (1993), etc.
- In a smooth cone in $\mathbb{R}^d$ with radially constant direction of reflection, Kwon and Williams (1991) characterize Reflecting Brownian Motion (RBM) as a solution of a submartingale problem and give necessary and sufficient conditions for existence and uniqueness of the solution that spends zero time at the vertex.
  C. and Kurtz (2022) gives a sufficient condition for the solution to be a semimartingale.
- In a convex polyhedron in $\mathbb{R}^d$, with constant direction of reflection on each face, Dai and Williams (1996) give sufficient conditions for existence and uniqueness of semimartingale RBM. The conditions are also necessary for a simple polyhedron.
- In a general piecewise $\mathcal{C}^1$ domain in $\mathbb{R}^d$, Dupuis and Ishii (1993) give sufficient conditions for strong existence and pathwise uniqueness of the solution to SDER.

Examples where none of the above results applies



$$|\text{angle}(g^1, n^1)| = |\text{angle}(g^2, n^2)| = \text{constant} \geq \frac{\pi}{4}$$

- The two examples from stochastic networks

RBM in a convex polyhedron

$$D := \bigcap_{i=1}^{m} D^i, \quad D^i \text{ a halfspace}, \quad N(x) := \left\{ n : (y-x)\cdot n \geq 0, \ \forall y \in \overline{D} \right\}$$

$g^i(x) \equiv g^i$, $G(x)$ closed, convex cone generated by $\{g^i, \, x \in \partial D^i\}$ b, σ constant

Theorem (Dai-Williams 1996)

a) *For every $x \in \partial D$, there exists $e \in N(x)$ such that*

$$e \cdot g > 0, \quad \forall g \in G(x) - \{0\}.$$

b) *For every $x \in \partial D$, there exists $v \in G(x)$ such that*

$$v \cdot n > 0, \quad \forall n \in N(x) - \{0\}.$$

Then, for every initial condition $X_0 \in \overline{D}$, there exists a solution of SDER and it is unique in distribution. The solution is a strong Markov process.

Remark

b) is a necessary condition for existence of RBM.
In dimension 2 and in a simple polyhedron, a) and b) are equivalent.

Piecewise smooth domains in $\mathbb{R}^2$

D bounded, connected, open set in $\mathbb{R}^2$

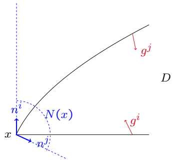
Define
$$N(x) := \left\{ n : \liminf_{y \in \overline{D},\, y \to x} \frac{(y-x)}{|y-x|} \cdot n \geq 0 \right\}.$$

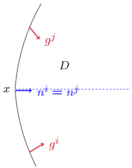
Assume D admits the following representation:
$$D = \bigcap_{i=1}^{m} D^i, \quad \partial D^i \in \mathcal{C}^1,$$
and, defining
$$I(x) := \{i : x \in \partial D^i\}, \quad x \in \partial D,$$
the set of ”corners” $\{x \in \partial D : |I(x)| > 1\}$ is finite.

$g^i(x)$ direction of reflection at $x \in \partial D^i$, $\quad \inf_{x \in \partial D^i} g^i(x) \cdot n^i(x) > 0.$

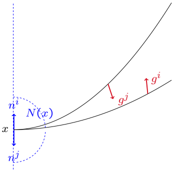
$G(x)$ the closed, convex cone generated by $\{g^i(x),\ i \in I(x)\}$

Cone points and cusp points





$N(x)$ does not contain any full straight line: cone case.



$N(x)$ contains a full straight line: cusp case.

In the cusp case we assume that the contact that ∂D^i and ∂D^j have between themselves is of order not higher than each of them has with their common tangent.

Piecewise smooth domains in $\mathbb{R}^2$: existence and uniqueness

g^i, b, σ Lischitz continuous, $\sigma(x)$ nonsingular at every corner $x \in \partial D$

Theorem (C. - Kurtz 2023)

a) *For every $x \in \partial D$, there exists $e \in N(x)$ such that*

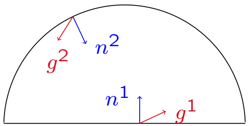
$$e \cdot g > 0, \quad \forall g \in G(x) - \{0\}.$$

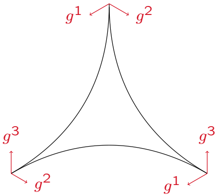
Then, for every initial condition $X_0 \in \overline{D}$, there exists a solution of SDER and it is unique in distribution. The solution is a strong Markov process.

Remark

In the case of RBM in a convex polygon with constant directions of reflection, our condition coincides with Dai and Williams': in this sense it is optimal.

Piecewise smooth domains in $\mathbb{R}^2$: examples





Piecewise smooth domains in $\mathbb{R}^2$: uniqueness

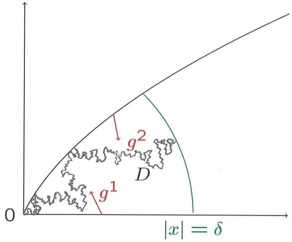
- In dimension 2, for any piecewise smooth domain D with a finite number of corners there is an open covering $\{U_k\}$ of $\overline{D}$ such that each U_k contains at most one corner x^k.
- Let D_k be a domain such that $\overline{D_k} \cap U_k = \overline{D} \cap U_k$ and such that ∂D_k is smooth except at x^k. By a localization result in C. - Kurtz (2023), if SDER has at most one (in distribution) solution in each $\overline{D_k}$ then it has at most one solution in $\overline{D}$.
- If x^k is a cone point, uniqueness in $\overline{D_k}$ has been proved in C. - Kurtz (2022). The result holds in $\mathbb{R}^d$.
- If x^k is a cusp point, uniqueness in $\overline{D_k}$ has been proved in C. - Kurtz (2018). The result can be extended to $\mathbb{R}^d$ for a class of cusps (work in progress).

Uniqueness for a domain with one singular point

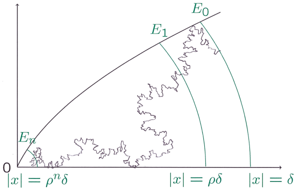
- it is enough to prove uniqueness among strong Markov solutions starting at 0 (C. - Kurtz (2019))
- it is enough to prove that any two strong Markov solutions starting at 0, X and $\widetilde{X}$, have the same hitting distributions:

$$\tau_\delta := \inf\{t \geq 0 : \ |X(t)| = \delta\}, \quad \widetilde{\tau}_\delta := \inf\{t \geq 0 : \ |\widetilde{X}(t)| = \delta\}$$

$$\tau_\delta, \widetilde{\tau}_\delta < \infty \ \ a.s. \quad \text{and} \quad \mathbb{P}(X(\tau_\delta) \in C) = \mathbb{P}(\widetilde{X}(\widetilde{\tau}_\delta) \in C) \quad \forall C, \forall \delta \leq \delta_0.$$



Uniqueness for a domain with one singular point



$$E_k := \{x \in \overline{D} : |x| = \rho^k \delta\}, \qquad \tau_k := \inf\{t \geq 0 : X(t) \in E_k\}$$

Intuition:

- $X, \widetilde{X}$ strong Markov solutions starting at 0; consider the Markov chains
$$\{\xi_h\} := \{X(\tau_{n-h})\}_{0 \leq h \leq n}, \qquad \{\widetilde{\xi}_h\} := \{\widetilde{X}(\widetilde{\tau}_{n-h})\}_{0 \leq h \leq n}$$
killed if $X, \widetilde{X}$ reach the origin before the next layer
- the transition kernels are the same, the difference between the two killed Markov chains is only in their initial distributions on E_n
- if the family of the transition kernels is "ergodic" $\Rightarrow$ the distributions of the two killed Markov chains after n steps, i.e. their distributions on E_0, as $n \to \infty$ should "forget" the initial distributions, hence be the same

A reverse ergodic theorem for inhomogeneous killed MCs

X, $\widetilde{X}$ two strong Markov solutions starting at 0. One can prove

$$\mathbb{E}\big[f(X(\tau_\delta))\big] = \frac{\int_{E_n} \big(Q_n \cdots Q_1 f\big)(x)\mu_n(dx)}{\int_{E_n} \big(Q_n \cdots Q_1 \mathbf{1}\big)(x)\mu_n(dx)}, \quad \mu_n(C) := \mathbb{P}(X(\tau_n) \in C),$$

$$\mathbb{E}\big[f(\widetilde{X}(\widetilde{\tau}_\delta))\big] = \frac{\int_{E_n} \big(Q_n \cdots Q_1 f\big)(x)\widetilde{\mu}_n(dx)}{\int_{E_n} \big(Q_n \cdots Q_1 \mathbf{1}\big)(x)\widetilde{\mu}_n(dx)}, \quad \widetilde{\mu}_n(C) := \mathbb{P}(\widetilde{X}(\widetilde{\tau}_n) \in C),$$

where

$$Q_k(x, C) := \mathbb{P}\big(\tau^x_{k-1} < \vartheta^x,\, X^x(\tau^x_{k-1}) \in C\big), \quad x \in E_k, \qquad C \subseteq E_{k-1}, \quad k \geq 1,$$

and X^x is a solution starting at x (uniquely determined up to $\vartheta^x := \inf\{t \geq 0 : X^x(t) = 0\}$), $\tau^x_{k-1} := \inf\{t \geq 0 : X^x(t) \in E_{k-1}\}$.

Goal:

$$\lim_{n\to\infty} \frac{\int_{E_n} \big(Q_n \cdots Q_1 f\big)(x)\ \mu_n(dx)}{\int_{E_n} \big(Q_n \cdots Q_1 \mathbf{1}\big)(x)\ \mu_n(dx)} \text{ is independent of } \{\mu_n\}$$

A reverse ergodic theorem for inhomogeneous killed MCs

Theorem (C. - Kurtz 2022)

$E_0, \dots E_n, \dots$ a sequence of compact metric spaces, Q_n a subprobability transition kernel from E_n to E_{n-1}

$f_{n,\widetilde{x}}(x,\cdot)$ the Radon-Nykodim derivative of $Q_n(x,\cdot)$ w.r.t. $\big(Q_n(x,\cdot) + Q_n(\widetilde{x},\cdot)\big)$

$$\epsilon_n(x,\widetilde{x}) := \int \big(f_{n,\widetilde{x}}(x,y) \wedge f_{n,x}(\widetilde{x},y)\big)\big(Q_n(x,dy) + Q_n(\widetilde{x},dy)\big), \quad x,\widetilde{x} \in E_n.$$

Assume Q_n is not identically zero and there exist $c_0 > 0$ and $\epsilon_0 > 0$ such that

(i) $\inf_n \inf_{x,\widetilde{x}\in E_n} \epsilon_n(x,\widetilde{x}) \geq \epsilon_0$,

(ii) $\inf_n \inf_{x,\widetilde{x}\in E_n} (Q_n \cdots Q_1)(x,E_0)/(Q_n \cdots Q_1)(\widetilde{x},E_0) \geq c_0$.

Then $\sup_{x\in E_n} Q_n \cdots Q_1 \mathbf{1}(x) > 0$ and, for every $f \in \mathcal{C}(E_0)$, $\{\mu_n\}$, $\mu_n \in \mathcal{P}(E_n)$, the limit

$$\lim_{n\to\infty} \frac{\int Q_n \cdots Q_1 f(x)\mu_n(dx)}{\int Q_n \cdots Q_1 \mathbf{1}(x)\mu_n(dx)}$$

exists and is independent of $\{\mu_n\}$.

Lower bound on $\epsilon_n(x, \widetilde{x})$: cone case

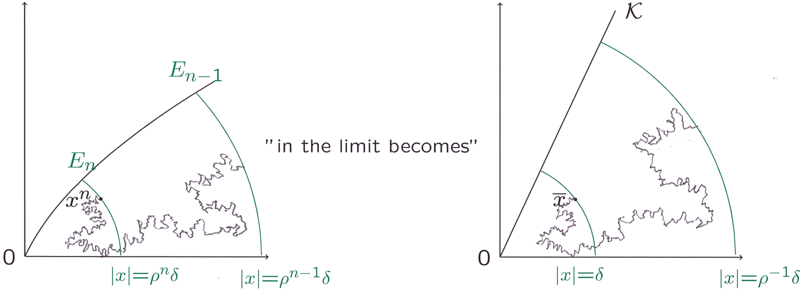
Let $x^n \in E_n$ be s.t. $\rho^{-n} x^n \to \bar{x}$. Then

$$\rho^{-n} X^{x^n}(\rho^{2n} \cdot) \xrightarrow{\mathcal{L}} \overline{X}^{\bar{x}},$$

where $\overline{X}$ is the Reflecting Brownian Motion in the "tangent cone"

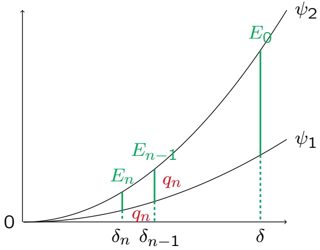
$$\mathcal{K} := \{x \in \mathbb{R}^2 : x \cdot n^1(0) > 0, \, x \cdot n^2(0) > 0\},$$

with directions of reflection $g^1(0)$, $g^2(0)$ and coefficients $b = 0$, $\sigma = \sigma(0)$.



Lower bound on $\epsilon_n(x, \widetilde{x})$: cusp case

Same argument as in the cone case, but different choice of $\{E_n\}$:



$$q_1 := \psi_2(\delta) - \psi_1(\delta), \quad \delta_1 := \delta - q_1,$$

$$q_n := \psi_2(\delta_{n-1}) - \psi_1(\delta_{n-1}), \quad \delta_n := \delta_{n-1} - q_n \quad (q_n^{-1}\delta_n \to \infty).$$

Lower bound on $\epsilon_n(x, \widetilde{x})$: cusp case

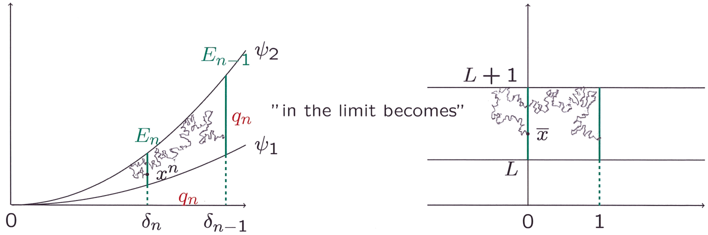
Let $x^n \in E_n$ be s.t. $q_n^{-1} x_2^n \to \bar{x}_2$ ($q_n^{-1} x_1^n = q_n^{-1} \delta_n \to \infty$). Then

$$q_n^{-1} \left(X_1^{x^n}(q_n^2 \cdot) - \delta_n, X_2^{x^n}(q_n^2 \cdot) \right) \xrightarrow{\mathcal{L}} \overline{X}^{\bar{x}},$$

where $\overline{X}$ is the Reflecting Brownian Motion in the strip

$$\{x \in \mathbb{R}^2 : L < x_2 < L + 1\}, \quad L := \lim_{x_1 \to 0^+} \frac{\psi_1(x_1)}{\psi_2(x_1) - \psi_1(x_1)},$$

with directions of reflection $g^1(0)$, $g^2(0)$ and coefficients $b = 0$, $\sigma = \sigma(0)$.



Domains with one singular point in $\mathbb{R}^d$: cone case

$$D \text{ bounded, connected, open set,} \quad \partial D - \{0\} \in \mathcal{C}^1.$$

There is an open cone $\mathcal{K}$,

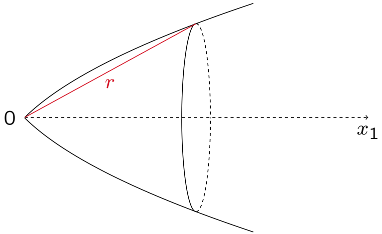
$$\mathcal{K} := \{rz,\ z \in \mathcal{S},\ r > 0\},\ \ \mathcal{S} \text{ a smooth domain in } S^{d-1},$$

such that, for some $r_D > 0$, $r \leq r_D$,

$$d_H(\overline{D} \cap \partial B_r(0), \overline{\mathcal{K}} \cap \partial B_r(0)) \leq c_D r^2, \quad d_H(\partial D \cap \partial B_r(0), \partial \mathcal{K} \cap \partial B_r(0)) \leq c_D r^2,$$

and for $|x| = r$, z the closest point to $\frac{x}{r}$ on $\partial \mathcal{S}$,

$$|n(x) - n^{\mathcal{K}}(z)| \leq c_D r.$$



Existence and uniqueness for the cone case

$$g(x) \text{ the direction of reflection at } x \in \partial D - \{0\}, \quad \inf_{x \in \partial D - \{0\}} g(x) \cdot n(x) > 0.$$

There is a smooth, unit vector field $\overline{g}$ on $\partial \mathcal{S}$ such that, for $x \in \partial D$, $|x| = r$, z the closest point to $\frac{x}{r}$ on $\partial \mathcal{S}$,

$$|g(x) - \overline{g}(z)| \leq c_g r, \qquad r \leq r_D.$$

$$\text{(in particular, for } x^n \in \partial D,\ |x^n| \to 0,\ \frac{x^n}{|x^n|} \to z, \quad g(x^n) \to \overline{g}(z))$$

$G(0)$ the closed, convex cone generated by $\{\overline{g}(\overline{z})), \overline{z} \in \partial \mathcal{S}\}$, $\sigma(0)$ nonsingular

Theorem (C. - Kurtz 2022)

a) *There exists $e \in N(0)$ such that*

$$e \cdot g > 0, \quad \forall g \in G(0) - \{0\}.$$

b)
$$G(0) \cap \mathcal{K} \neq \emptyset.$$

then, for every initial condition $X_0 \in \overline{D}$, there exists a solution of SDER and it is unique in distribution. The solution is a strong Markov process.

Domains with one singular point in $\mathbb{R}^d$: cusp case

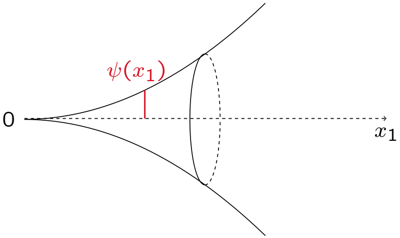
$$D \text{ bounded, connected open set,} \quad \partial D - \{0\} \in \mathcal{C}^1$$

There are a domain $D_0 \subseteq \mathbb{R}^{d-1}$ and a function $\psi : [0,\infty) \to [0,\infty)$, such that, for some $c_D > 0$,

$$D \cap \{x_1 \leq c_D\} = \{x = (x_1, x^{d-1}),\, x^{d-1} \in \mathbb{R}^{d-1} :\ 0 < x_1 \leq c_D,\ \frac{x^{d-1}}{\psi(x_1)} \in D_0\}$$

$$D_0 \text{ bounded,} \quad 0 \in D_0, \quad \partial D_0 \text{ smooth,}$$

$$\psi \in \mathcal{C}^1[0,\infty), \quad \psi(0) = \psi'(0) = 0, \quad \psi(t) > 0 \text{ for } t > 0,$$



Existence and uniqueness for the cusp case

$$g(x) \text{ the direction of reflection at } x \in \partial D - \{0\}, \quad \inf_{x \in \partial D - \{0\}} g(x) \cdot n(x) > 0.$$

There is a smooth, unit vector field $\overline{g}$ on ∂D_0 such that, for $x \in \partial D$,

$$|g(x) - \overline{g}(\frac{x^{d-1}}{\psi(x_1)})| \leq c_g x_1, \qquad x_1 \leq c_D,$$

$$\text{(in particular, for } x^n \in \partial D,\ |x^n| \to 0,\ \frac{(x^n)^{d-1}}{\psi(x_1^n)} \to z, \quad g(x^n) \to \overline{g}(z))$$

$G(0)$ the closed,convex cone generated by $\{\overline{g}(z)), z \in \partial D_0\}, \sigma(0)$ nonsingular

Theorem (work in progress)

a) *There exists $e \in N(0)$ such that*

$$e \cdot g > 0, \quad \forall g \in G(0) - \{0\}.$$

Then, for every initial condition $X_0 \in \overline{D}$, there exists a solution of SDER and it is unique in distribution. The solution is a strong Markov process.

Existence for piecewise smooth domains in $\mathbb{R}^d$

D bounded, connected, open set in $\mathbb{R}^d$

Assume D admits a representation $D = \bigcap_{i=1}^m D^i$, $\partial D^i \in \mathcal{C}^1$. such that

$$N(x) = \{ \sum_{i \in I(x)} \eta_i n^i(x),\ \eta_i \geq 0\}, \quad \forall x \in \partial D.$$

For $y \in D^c$ let

$$I(y) := \{i : \ y \notin D^i\},$$

For every $x \in \partial D$ there is $\delta(x) > 0$ such that $I(y) \subseteq I(x)$ for $y \in B_{\delta(x)}(x)$.

For $x \in \partial D$, define the following family of subsets of $I(x)$

$$\mathcal{I}(x) := \left\{ I \subseteq I(x) : \ I = I(y) \text{ for some } y \in \overline{D}^c \cap B_{\delta(x)}(x) \right\},$$

and the subcones

$$N^I(x) := \{\sum_{i \in I} \eta_i n^i(x),\ \eta_i \geq 0\}, \quad G^I(x) := \{\sum_{i \in I} \eta_i g^i(x),\ \eta_i \geq 0\}, \quad I \in \mathcal{I}(x).$$

Existence for piecewise smooth domains in $\mathbb{R}^d$

Theorem (C. - Kurtz 2019)

b, σ, g^i continuous, $\inf_{x\in\partial D^i} g^i(x)\cdot n^i(x) > 0$

a) *for every $x \in \partial D$, there exists $e \in N(x)$ such that*

$$e \cdot g > 0, \quad \forall g \in G(x) - \{0\}$$

b) *for every $x \in \partial D$, for every $I \in \mathcal{I}(x)$, $N^I(x)$ does not contain any full straight line and for every $n \in N^I(x) - \{0\}$ there is $g \in G^I(x)$ such that*

$$n \cdot g > 0$$

Then, for every initial condition $X_0 \in \overline{D}$, there exists a strong Markov solution of SDER.
If uniqueness in distribution holds among strong Markov solutions of SDER then it holds among all solutions.

Remark

In the case of a simple, convex polyhedron, our conditions are equivalent to Dai and Williams'.

Existence for piecewise smooth domains in $\mathbb{R}^d$

Keypoints of proof

- X is a solution of SDER if and only if X is a natural solution of the corresponding constrained martingale problem (introduced by Kurtz (1987) and (1989))
- One can construct a natural solution of the constrained martingale problem by a limiting procedure without proving oscillation estimates.

One can formulate constrained martingale problems for all sorts of boundary behavior (Wentzell boundary conditions, jumps from the boundary, etc.)

Constrained martingale problem

$$Af(x) := \nabla f(x) \cdot b(x) + \frac{1}{2}\mathrm{tr}\big(\sigma(x)\sigma^T(x)D^2 f(x)\big)$$

$$\Xi := \{(x,u) \in \partial D \times \mathbb{R}^d : u \in G(x),\ |u| = 1\}, \quad Bf(x,u) := \nabla f(x) \cdot u$$

Constrained martingale problem (Kurtz 1987, 1989. C.- Kurtz 2019)

X is a solution of the *constrained martingale problem* for (A, D, B, Ξ) if there exists a random measure Λ on $[0,\infty) \times \Xi$ and a filtration $\{\mathcal{F}_t\}$ such that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_{[0,t]\times\Xi} Bf(x,u)\Lambda(ds \times dx \times du)$$

is a $\{\mathcal{F}_t\}$-local martingale. X is a *natural solution* if

$$X(t) = Y(\lambda_0^{-1}(t)), \quad \Lambda([0,t] \times C) = \int_{[0,\lambda_0^{-1}(t)]\times S_1(0)} \mathbf{1}_C(Y(s),u)\Lambda_1(ds \times du),$$

where $(Y, \lambda_0, \Lambda_1)$ is a solution of the *controlled martingale problem* for (A, D, B, Ξ) (a slowed down martingale problem).

Thank you for your attention!

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