CLT for reflecting Brownian motion in generalized parabolic domains

Isao Sauzedde[1] (joint work with A. Mijatovic[1,2] and A. Wade[3])

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EPSRC Anomalous diffusion via self-interaction and reflection

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Framework

We consider a standard Brownian motion with oblique reflection

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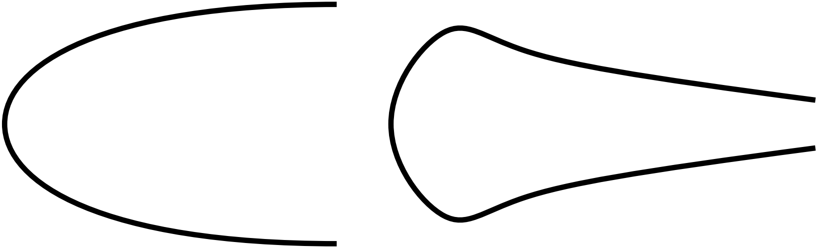
in the domain

$$\mathcal{D} := \{(x,y) \in \mathbb{R}_+ \times \mathbb{R}^d : |y|_d < b(x)\}, \qquad b(x) \propto_{x\to\infty} x^\beta.$$



$\beta > 0$ $\qquad$ $\beta < 0$

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Asymptotic reflection vector field:

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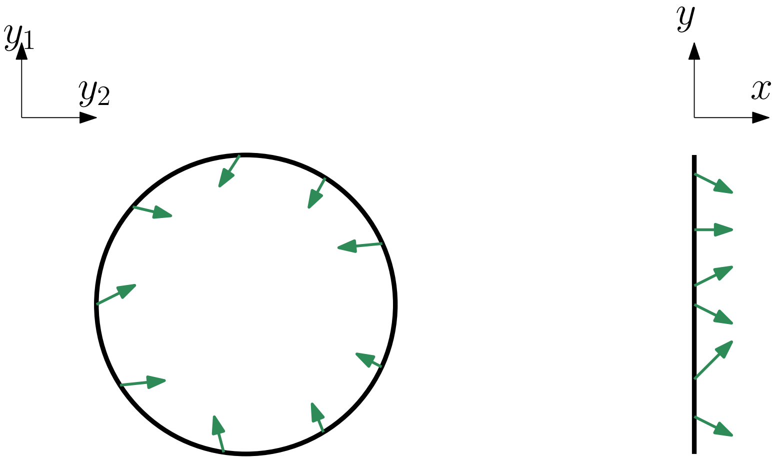
- For all $\theta \in \mathbb{S}_1^{d-1}$, $V_{x,b(x)\theta}$ admits a limit V_θ^∞ as $x \to \infty$.
- With the proper normalisation of V, V_θ^∞ has a constant positive component in the x direction.



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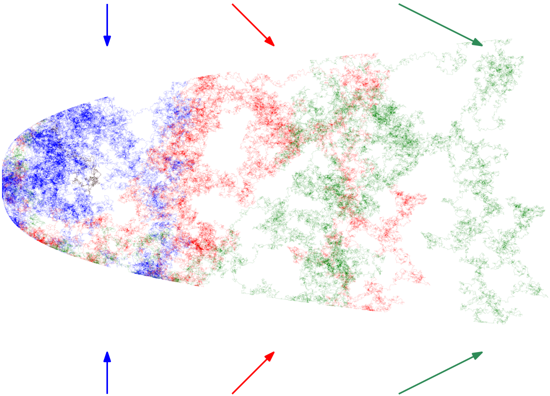

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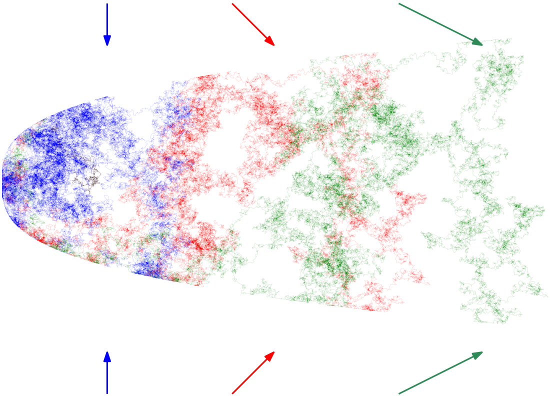

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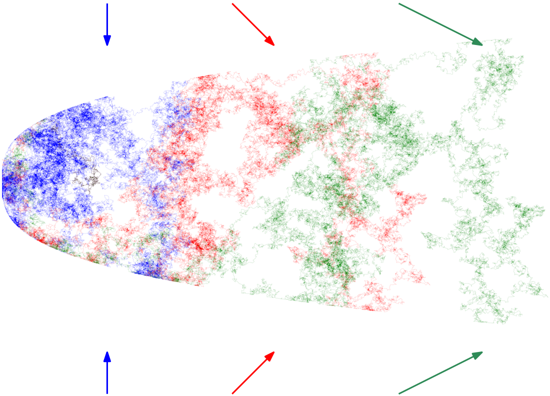

- Can we go farther and get a second order estimation?
- What about Y?

Consider the Brownian motion $Z' = (X', Y')$ on the unit cylinder, reflected according to V^∞:

$$\mathrm{d}Z'_t = \mathrm{d}B_t + V^\infty_{Y'_t}\, \mathrm{d}L^{Z'}_t,$$

with $X'_0 = 0$ and Y'_0 started from the invariant measure for Y'.

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with $X'_0 = 0$ and Y'_0 started from the invariant measure for Y'.
Consider also $Y_{T,t} = \frac{Y_{T+b(X_T)^2 t}}{b(X_T)}$ and $X_{T,t} = \frac{X_{T+b(X_T)^2 t} - X_T}{b(X_T)}$.

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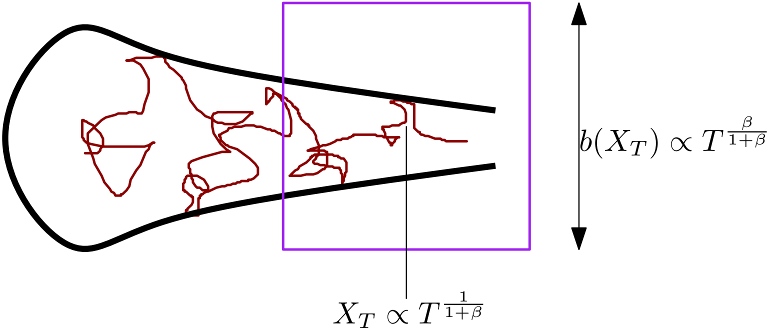

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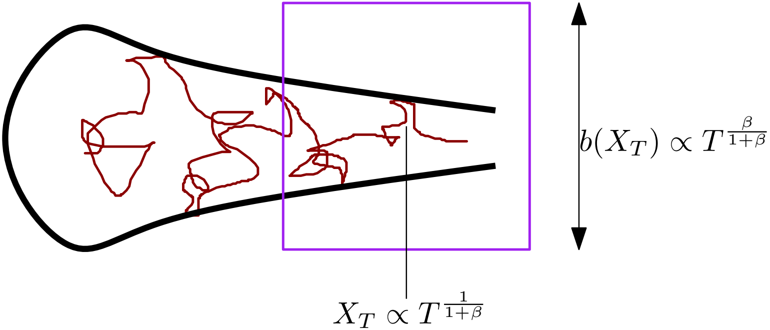

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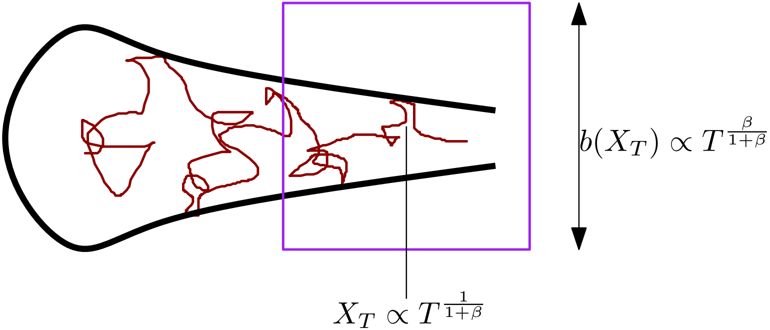

Then, as $T \to \infty$, $(X_{T,t}, Y_{T,t})$ converges in distribution toward Z'.
In particular, $\frac{Y_T}{b(X_T)}$ converges in distribution.

Furthermore, if $\beta \in (-\frac{1}{3}, \frac{1}{3})$,

$$\frac{X_T - cT^{\frac{1}{1+\beta}}}{\sqrt{T}} \xrightarrow[T\to\infty]{(d)} \mathcal{N}(0, c').$$

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converges almost surely.

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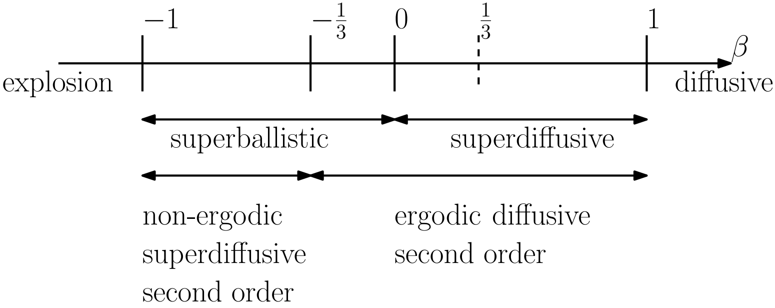
Toy model: $\mathrm{d}X_t = \mathrm{d}B_t + \frac{1}{X_t^\beta}\,\mathrm{d}t.$



-1
$-\frac{1}{3}$
0
$\frac{1}{3}$
1
β
explosion
diffusive
superballistic
superdiffusive
non-ergodic
superdiffusive
second order
ergodic diffusive
second order

Thanks for your attention!