A Dirichlet Process Characterization of RBM with Drift in a Wedge

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40 Years of RBM and Related Topics

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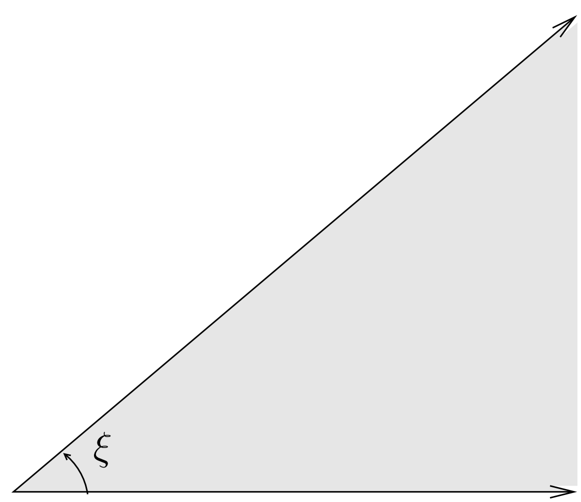
Motivation

- Let S be a wedge in $\mathbb{R}^2$ whose state space in polar coordinates is given by

$$S = \{(r, \theta) : r \geq 0, 0 \leq \theta \leq \xi\}$$

for some $0 < \xi < 2\pi$.



Motivation

- Consider the problem of defining a process Z that behaves as a standard Brownian motion with constnat drift in the interior of S and is reflected back into S when hitting its boundary.

- The method of reflection is defined as follows.
- Let
$$\partial S_1 = \{(r, \theta) : r > 0, \theta = 0\}$$
denote the lower boundary of the wedge, and let
$$\partial S_2 = \{(r, \theta) : r > 0, \theta = \xi\}$$
denote its upper boundary.

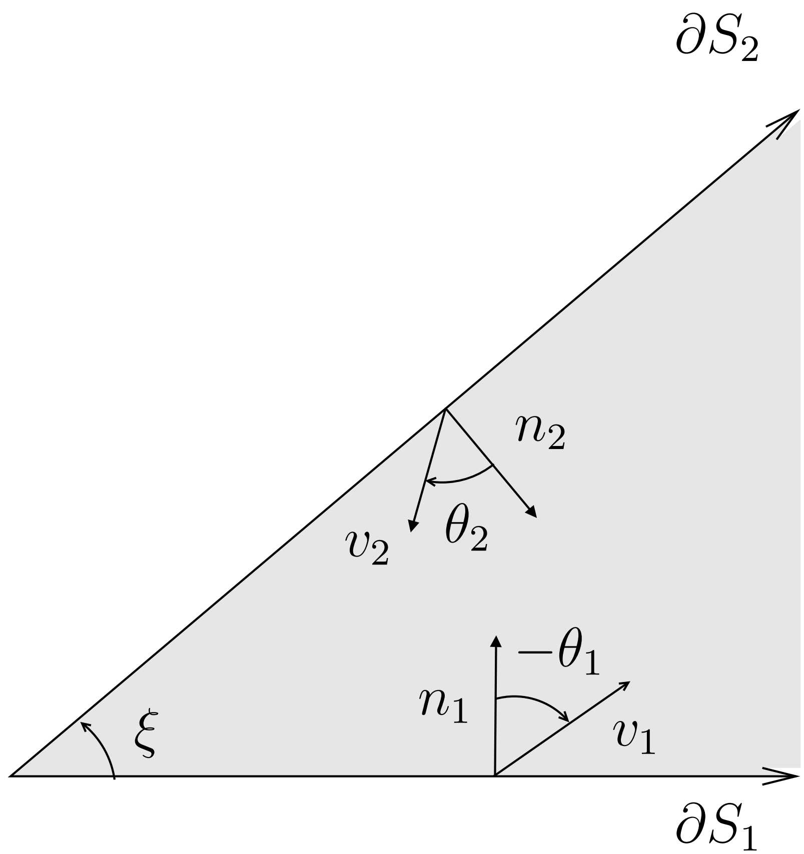
Motivation

- The angle of reflection off of the edge ∂S_1 is denoted by $-\pi/2 < \theta_1 < \pi/2$, and the angle of reflection off of the edge $\partial S_2 = \{(r, \theta) : r > 0, \theta = \xi\}$ is denoted by $-\pi/2 < \theta_2 < \pi/2$.

- Both of these angles are measured with respect to the inward facing normal off their respective edge, with angles directed toward the origin assumed to be positive.

- We denote by n_1 the inward facing unit normal off of the edge ∂S_1, and by v_1 the unit vector of reflection off of ∂S_1. Similar quantities are defined for ∂S_2.

Motivation



Motivation

- For some parameter values, the process Z may be defined as the unique solution to a stochastic differential equation with reflection.
- This is not true however for all parameter values, and in particular for those in which in which the resulting process Z s not a semimartingale.
- Nevertheless, a process Z with the desired characteristics was rigorously defined by Varadhan and Williams (1985) for all paramter combinations as the solution to the submartingale problem in the driftless case.
- In this work, we extend the results of Varadhan and Williams (1985) to include a constant drift term.
- We also provide a result on the sample path properties of the process Z in the non-semimartingale case.
- The motivation for this comes from studying queueing systmes where the approximating queue length process is not a semimartingle.

Outline of Talk

- Definition of the Submartingale Problem with Drift
- The Quantity α
- Existence and Uniqueness Result
- Coupled Processor Model Example
- Hitting Time of Vertex
- Dirichlet Process and p-variation Result
- Extended Skorokhod Problem Result

Definition of the Submartingale Problem with Drift

- Let C_S denote the space of continuous functions with domain $\mathbb{R}_+ = [0, \infty)$ and range S.

- For each $t \geq 0$ and $\omega \in C_S$, denote by $Z(t) : C_S \mapsto S$ the coordinate map $Z(t)(\omega) = Z(t, \omega) = \omega(t)$, and also define the coordinate mapping process $Z = \{Z(t), t \geq 0\}$.

- Then, for each $t \geq 0$, set $\mathcal{M}_t = \sigma\{Z(s), 0 \leq s \leq t\}$, and set $\mathcal{M} = \sigma\{Z(s), s \geq 0\}$.

- Next, for each $n \geq 1$ and $F \subset \mathbb{R}^2$, denote by $C^n(F)$ the set of n-times continuously differentiable functions in some domain containing F, and let C_b^n be the set of functions in $C^n(F)$ that have bounded partial derivatives up to and including order n on F.

- Finally, define the differential operators $D_j = v_j \cdot \nabla$ for $j = 1, 2$, and denote by $\triangle$ the Laplacian operator.

Definition of the Submartingale Problem with Drift

Definition 1. *A family of probability measures* $\{P_\mu^z, z \in S\}$ *on* $(C_S, \mathcal{M})$ *is said to solve the submartingale problem with drift* $\mu \in \mathbb{R}_+^2$ *if for each* $z \in S$*, the following 3 conditions hold,*

1. $P_\mu^z(Z(0) = z) \; = \; 1,$
2. *For each* $f \in C_b^2(S)$*, the process*

$$\left\{ f(Z(t)) - \int_0^t \mu \cdot \nabla f(Z(s))ds - \frac{1}{2}\int_0^t \triangle f(Z(s))ds, \quad t \geq 0 \right\}$$

   is a submartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, P_\mu^z)$ *whenever* f *is constant in a neighborhood of the origin and satisfies* $D_i f \geq 0$ *on* ∂S_i *for* $i = 1, 2,$

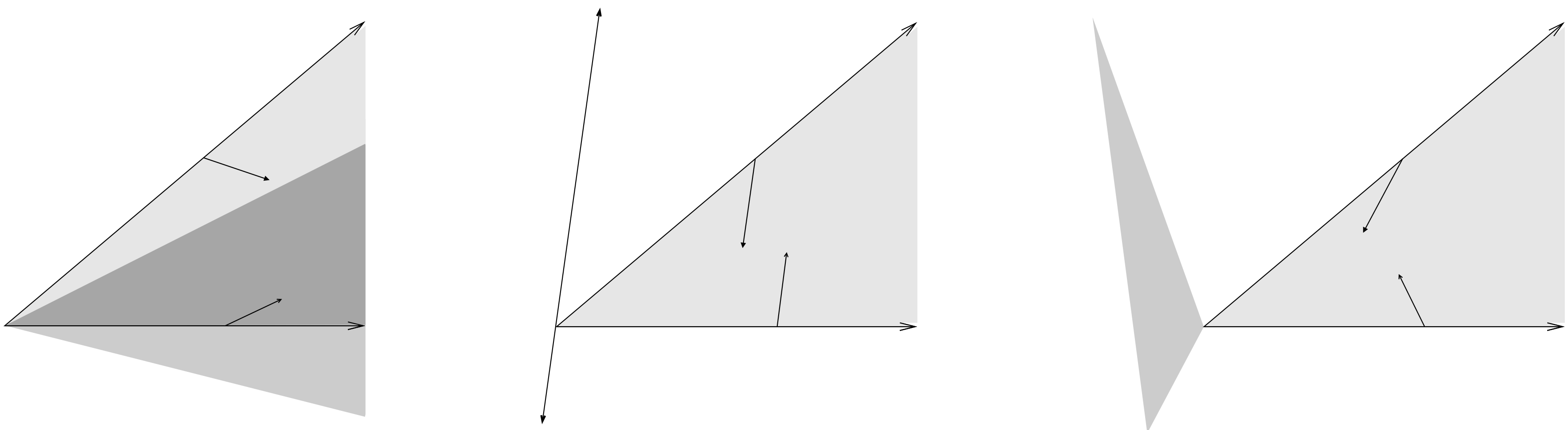
3.

$$E_\mu^z\left[\int_0^\infty 1\{Z(t) = 0\}dt\right] \; = \; 0.$$

The Quantity α

- Let

$$\alpha = \frac{\theta_1 + \theta_2}{\xi}.$$



$\alpha < 1$ $\alpha = 1$ $\alpha > 1$

- Much of the behavior of Z is determined by the value of α.

The Quantity α

- As we shall see, several characteristics of the solution to the submartigale problem are determined by the value of α.
- In the driftless case, Varadhan and Williams (1985) proved that if $\alpha < 2$, then there exists a unique solution to the submartingale probelm.
- They also showed that if $\alpha \geq 2$, then there is a solution satisfying only Conditions 1 and 2 of the submartingale problem. In this case, the process Z is absorbed at the origin upon reaching it.
- The results of Williams (1985) show that Z is a semimartingale if and only if $\alpha < 1$ or $\alpha \geq 2$.
- The results of Kang and Ramanan (2010) imply that Z is a Dirichlet process if $\alpha = 1$.

Existence and Uniqueness

Theorem 1. *If $\alpha < 2$, then for each $\mu \in \mathbb{R}^2$ there exists a unique solution $\{P_\mu^z, z \in S\}$ to the submartingale problem with drift. Moreover, there exists a process X defined on $(C_S, \mathcal{M}, \mathcal{M}_t)$ which, for each $z \in S$, is a 2-dimensional Brownian motion with drift μ started at z under P_μ^z.*

- In the driftless case, the process X is approximated from inside the wedge using the process Z and then an L^2 limit is taken.

- A Girsanov transformation is then used to prove existence for the case of non-zero drift.

The Coupled Processor Model

- The coupled processor model (Fayolle and Iasnogorodski (1979), Boxma and Cohen (2000)) is as an example where the queue length process is conjectured in heavy traffic to be approximated by a non-semimartingale Brownian motion in an orthant.

- Two servers labeled $i = 1, 2$.

- Customers arrive to server i according to a Poisson process with rate λ_i, where the two arrival processes are independent of one another.

- The service times at server i are i.i.d. with CDF B_i and mean $1/\mu_i$.

- If both servers are busy serving a customer, then each server i works at rate 1.

The Coupled Processor Model

- On other hand, if server 2 is idle, then server 1 works at rate ρ_1, and similarly if server 1 is idle, then server 2 works at rate ρ_2.

- In this case, the conjectured approximating RBM has parameters $\xi = \pi/2$ and $v_1 = (-\mu_1(\rho_1 - 1), \mu_2)$ and $v_2 = (\mu_1, -\mu_2(\rho_2 - 1),)$.

- It can then be shown that the case of $1 < \alpha < 2$ corresponds to

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} < 1.$$

Hitting Time of the Vertex

- Let $\tau_0 = \inf\{t \geq 0 : Z(t) = 0\}$ be the stopping time with respect to $\{\mathcal{M}_t, t \geq 0\}$ representing the first time that Z reaches the vertex of the wedge.

- It is known (Varadhan and Williams (1985)) that in the driftless case if $\alpha > 0$, then $P_0^z(\tau_0 < \infty) = 1$ for each $z \in \mathbb{S}$, and $P_0^z(\tau_0 < \infty) = 0$ otherwise $(\alpha \leq 0)$.

Hitting Time of the Vertex

Theorem 2. *If* $\alpha \geq 1$, *then*

$$P^z_\mu(\tau_0 < \infty) > 0 \text{ for each } \mu \in \mathbb{R}^2 \text{ and } z \in S.$$

Moreover, if in addition to the $\alpha \geq 1$ *condition we also have that*

$$co(v_1, v_2, \mu) \cap S = \{0\}, \tag{1}$$

then for each $z \in S$,

$$P^z_\mu(\tau_0 < \infty) = 1.$$

- In the case of $\alpha > 1$, condition (1) is equivalent to the requirement the vector $R^{-1}\mu$ has at least one non-negative component.

Dirichlet Process Definition

Definition 2. *Let $z \in S$. A continuous process Y defined on $(C_S, \mathcal{F}, \mathcal{F}_t, P_\mu^z)$ is said to be of zero energy if for each $T > 0$,*

$$\sum_{t_i \in \pi^n} \|Y(t_i) - Y(t_{i-1})\|^2 \xrightarrow{P} 0 \text{ as } n \to \infty, \tag{2}$$

for any sequence $\{\pi^n, n \geq 1\}$ of partitions of $[0, T]$ with $\|\pi^n\| \to 0$ as $n \to \infty$.

- The notion of a zero-energy process can be used to define a Dirichlet process.

Dirichlet Process Definition

Definition 3. *Let $z \in S$. The stochastic process Z is said to be a Dirichlet process on $(C_S, \mathcal{F}, \mathcal{F}_t, P_\mu^z)$ if we may write*

$$Z = X + Y, \tag{3}$$

where X is an $\mathcal{F}_t$-adapted local martingale and Y is a continuous, $\mathcal{F}_t$-adapted zero energy process with $Y(0) = 0$.

- The class of Dirichlet processes is larger than the class of semi-martingales.
- Moreover, there exists a stochastic calculus for the class of Dirichlet processes, including a change-of-variable formula.
- It is also worth noting that the decomposition $Z = X + Y$ appearing in Definition 3 may be shown to be unique.

Dirichlet Processes Result

Theorem 3. *Suppose that* $1 < \alpha < 2$. *Then,* Z *has the decomposition*

$$Z = X + Y,$$

where (X, Y) *is a pair of processes on* $(C_S, \mathcal{F}, \mathcal{F}_t)$ *such that for each* $z \in S$, X *is a Brownian motion with drift* μ *started from* z *and* Y *is a process of zero energy on* $(C_S, \mathcal{F}, \mathcal{F}_t, P^z_\mu)$. *In particular, for each* $z \in S$, *the process* Z *is a Dirichlet process on* $(C_S, \mathcal{F}, \mathcal{F}_t, P^z_\mu)$.

- The pair of processes (X, Y) appearing in Theorem 3 do not depend on $z \in S$. That is, there is a single pair of processes (X, Y) on $(C_S, \mathcal{F}, \mathcal{F}_t)$ such that the statement of the above theorem hold for each $z \in S$.

p-variation Definition

Definition 4. *Let $f : \mathbb{R}_+ \mapsto \mathbb{R}^d$ and $p > 0$. Then, f is said to be of finite strong p-variation if for each $T \geq 0$,*

$$V_p(f,T) \; = \; \sup \left\{ \sum_{t_i \in \pi} \|f(t_i) - f(t_{i-1})\|^p : \pi \in \pi(T) \right\} \; < \; \infty,$$

where the supremum is taken over all partitions π of $[0,T]$.

- Strong p-variation is a generalization of the concept of bounded variation.
- If $V_p(f,T) < \infty$, then $V_q(f,T) < \infty$ for $q \geq p$.

p-variation Result

Theorem 4. *Suppose that* $1 < \alpha < 2$*. Then, for each* $p > \alpha$ *and* $z \in S$*,*

$$P_\mu^z(V_p(Y, [0, T]) < +\infty) \; = \; 1, \; T \geq 0.$$

Furthermore, for each $0 < p \leq \alpha$*,*

$$P_\mu^0(V_p(Y, [0, T]) < +\infty) \; = \; 0, \; T \geq 0.$$

- Note that the sample paths of Y become rougher as α increases from 1 to 2.

The Extended Skorokhod Problem

- Next, we provide a rigorous statement of the fact that Z is the reflected version of the Brownian motion X.
- Let $D(\mathbb{R}_+, \mathbb{R}^d)$ denote the space of $\mathbb{R}^d$-valued functions, with domain $\mathbb{R}_+$, that are right-continuous with left limits.
- Also, let $D_S(\mathbb{R}_+, \mathbb{R}^2)$ be the set of $f \in D(\mathbb{R}_+, \mathbb{R}^2)$ such that $f(0) \in S$.
- Next, let $d(\cdot)$ be a set-valued mapping defined on S such that $d(z)$ is a closed convex cone in $\mathbb{R}^2$ for every $z \in S$. In particular, we define

$$d(z) = \begin{cases} \{\alpha v_1, \alpha \geq 0\}, & \text{for } z \in \partial S_1, \\ \{\alpha v_2, \alpha \geq 0\}, & \text{for } z \in \partial S_2, \\ V, & \text{for } z = 0, \\ \{0\}, & \text{for } z \in \text{Int}(S). \end{cases} \tag{4}$$

The Extended Skorokhod Problem

Definition 5. *[Ramanan (2006)] The pair of processes* $(\phi, \eta) \in D_S(\mathbb{R}_+, \mathbb{R}^2) \times D(\mathbb{R}_+, \mathbb{R}^2)$ *solve the ESP* $(S, d(\cdot))$ *for* $\psi \in D(\mathbb{R}_+, \mathbb{R}^2)$ *if* $\phi(0) = \psi(0)$*, and if for all* $t \in \mathbb{R}_+$*, the following properties hold,*

1. $\phi(t) \;=\; \psi(t) + \eta(t),$
2. $\phi(t) \in S,$
3. *For every* $s \in [0, t]$,
$$\eta(t) - \eta(s) \;\in\; \overline{\mathrm{Conv}}[\cup_{u \in (s,t]} d(\phi(u))],$$
4. $\eta(t) - \eta(t-) \in \overline{\mathrm{Conv}}[d(\phi(t))].$

- The key difference between the extended Skorokhod problem and the standard Skorokhod is that the pushing function η is no longer required to be of bounded variation.

Theorem 5. *Suppose that $1 < \alpha < 2$. Then, for each $z \in S$, the ESP $(S, d(\cdot))$ for the Brownian motion X with drift μ on $(C_S, \mathcal{F}, \mathcal{F}_t, P_\mu^z)$ has a solution P_μ^z-a.s. if and only if*

$$\overline{\text{Conv}}(V \cup \{\alpha v_1, \alpha \geq 0\} \cup \{\alpha v_2, \alpha \geq 0\}) = \mathbb{R}^2. \tag{5}$$

In this case, (Z, Y) solves the ESP $(S, d(\cdot))$ for X.

- By trivially setting $V = \mathbb{R}^2$, it follows that one may always find a V such that (5) holds. However, using the fact that $1 < \alpha < 2$, it is straightforward to verify that the smaller set $V = \{\alpha v_0, \alpha \geq 0\}$ for any v_0 in the interior of S satisfies (5) as well.

- Note also that Theorem 5 does not claim that (Z, Y) is the unique solution to the ESP $(S, d(\cdot))$ for X.

Proof of Theorem 4

- Proof proceeds sample-pathwise and, more specifically, uses an excursion based approach.
- Let $\Lambda = \{t \in \mathbb{R}_+ : Z(t) = 0\}$ be the zero set of Z.
- Then,

$$\Lambda^C \;=\; [0, \tau_0) \cup \left(\bigcup_{i=1}^{\infty} (G_i, D_i) \right),$$

where τ_0 is the first hitting time of the origin by Z.

- We refer to (G_i, D_i) as the ith excursion of Z from the vertex.
- Define the inverse local time process

$$L^{-1}(a) \;=\; \inf\{t \geq 0 : L(t) > a\}, \quad a \geq 0.$$

- Then, $L^{-1}(\mathbb{R}_+) \subset \Lambda$ and .

$$\Lambda \backslash L^{-1}(\mathbb{R}_+) \;=\; \bigcup_{i=1}^{\infty} \{G_i\}.$$

Proof of Theorem 4

- Since $Z \circ L^{-1} = 0$, it follows that $Y \circ L^{-1} = -X \circ L^{-1}$.

- On the other hand, X has sample that paths that are a.s. of finite strong p-variation for $p > 2$.

- Moreover, it is known (Williams (1987)) that under P^0, L^{-1} is a stable subordinator of index $\alpha/2$, which a.s. has sample paths that are of finite strong p-variation for $p > \alpha/2$.

- It then follows by definition that $Y \circ L^{-1} = -X \circ L^{-1}$ has finite p-variation a.s. for $p > \alpha$.

- This implies that a.s. the sample paths of Y have finite p-variation on $L^{-1}(\mathbb{R}_+)$ for $p > \alpha$.

Proof of Theorem 4

- It remains to show that Y has finite p variation on $\mathbb{R}_+ \backslash L^{-1}(\mathbb{R}_+)$.

- This involves studying the behavior of Y over each excursion away from the vertex of the wedge.

- However, it can be shown that Y satisfies the ordinary Skorkohod problem on each excursion and hence has finite 1-variation on excursion intervals.

THANK YOU!