Degenerate, competing three particle systems

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Introduction to rank-based diffusions

On $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, let us consider an n-dimensional diffusion $X(t) := (X_1(t), \ldots, X_n(t))$ described by

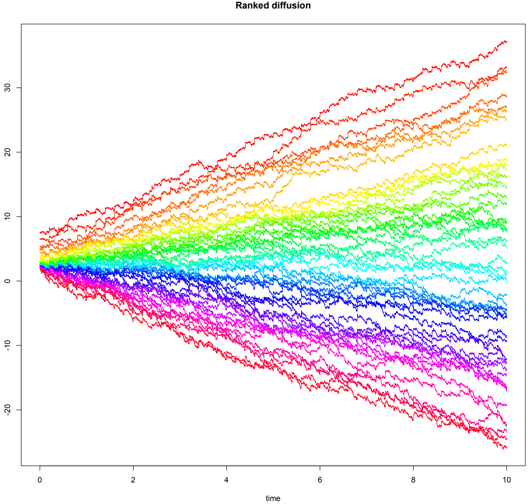
$$\mathrm{d}X_i(t) = \sum_{k=1}^{n} g_k \cdot 1_{\{X_i(t) = R_k(t)\}} \mathrm{d}t + \sum_{k=1}^{n} \sigma_k \cdot 1_{\{X_i(t) = R_k(t)\}} \mathrm{d}W_i(t)$$

for $t \geq 0$, $1 \leq i \leq n$ with $X(0) = \mathrm{x} \in \mathbb{R}^n$,
where $g_1, \ldots, g_n$ and $\sigma_1, \ldots, \sigma_n$ are some constants, $W(\cdot) := (W_1(\cdot), \ldots, W_n(\cdot))$ is an n-dimensional Brownian motion, $1_{\cdot}$ is the indicator function of sets and $R_k(t)$ is the k-th largest (reversed order statistics) among $(X_1(t), \ldots, X_n(t))$, i.e., $R_1(t) \geq \ldots \geq R_n(t)$ for every $t \geq 0$.
Here, we resolve the ties of ranking in favor of the lowest index, and we consider solution with the non-stickiness conditions

$$\int_0^\infty 1_{\{R_k(t) = R_{k+1}(t)\}} \mathrm{d}t = 0, \quad k = 1, 2, \ldots, n-1.$$

Simulated ranked process $R_k(\cdot)$

Figure: $g_k := 0.1 \cdot k - \overline{g}\,,\ k = 1, \ldots, n\,,\ \overline{g} := n(n+1)/20\,,$
$\sigma_k = 1 + 0.01 \cdot k\,,\ n = 50\,.$



With piece-wise constant functions

$$\mathrm{g}(x) := \sum_{k=1}^{n} g_k \cdot \chi_{n,k}(x)\,, \quad \boldsymbol{\sigma}(x) := \sum_{k=1}^{n} \sigma_k \cdot \chi_{n,k}(x)\,,$$

$\chi_{n,k}(x) := 1_{\{(n-k)/n < x \le (n-k+1)/n\}}\,, \; k = 1, \ldots, n\,, \; x \in \mathbb{R}$
and the empirical measure process

$$\boldsymbol{\rho}_n(t) := \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\delta}_{X_i(t)}\,, \quad t \ge 0\,,$$

the system can be rewritten as

$$\mathrm{d}X_i(t) = \mathrm{g}(F(X_i(t); \boldsymbol{\rho}_n(t)))\mathrm{d}t + \boldsymbol{\sigma}(F(X_i(t); \boldsymbol{\rho}_n(t)))\mathrm{d}W_i(t)$$

for $t \ge 0$, where $F(x; \boldsymbol{\mu}) := \boldsymbol{\mu}((-\infty, x))$, $x \in \mathbb{R}$ represents the cumulative distribution function of a probability measure $\boldsymbol{\mu}(\cdot)$, and $\boldsymbol{\delta}_x(\cdot)$ is the Dirac delta measure at $x \in \mathbb{R}$.

Under appropriate conditions, as $n \to \infty$, the empirical measure $\rho_n(\cdot)$ converges weakly to a deterministic path $\rho_\infty(\cdot)$, the unique solution of a McKean-Vlasov equation, and its cumulative distribution $u(t,x) := F(x; \rho_\infty(t))$, $t \geq 0$, $x \in \mathbb{R}$ satisfies the porous medium equation

$$\partial_t u = \partial_x(\mathrm{G}(u)) + \partial^2_{xx}(\mathrm{S}(u));$$

$$\mathrm{G}(\cdot) := -\int_0^{\cdot} \mathrm{g}(y)\mathrm{d}y, \quad \mathrm{S}(\cdot) := \frac{1}{2}\int_0^{\cdot} \boldsymbol{\sigma}^2(y)\mathrm{d}y.$$

(Dembo, Shkolnikov, Varadhan & Zeitouni ('12))

- The convergence is exponentially fast and the fluctuations around this limit are gaussian described by an SPDE (Kolli & Shkolnikov ('18)).
- Application to financial markets

Banner, Fernholz & Karatzas ('05) Ichiba et al. ('11) Jourdain & Reygner ('15), ...

Non-degenerate case

When $\sigma_k > 0$, $k = 1, \ldots, n$,

- Existence of weak solution $(\Omega, \mathcal{F}, \mathbb{P})$, $(X(\cdot), W(\cdot))$, by the theory of the martingale problem of Stroock & Varadhan with the Alexandroff-Krylov estimates,
- Uniqueness in distribution - by Bass & Pardoux ('87)
- Pathwise uniqueness

  - holds up to the time τ of triple collision

$$\tau := \inf\{t > 0 : X_i(t) = X_j(t) = X_k(t)$$

$$\text{for some } i \neq j, j \neq k, k \neq i\},$$

Prokaj ('11), Ichiba, Karatzas & Shkolnikov ('13), Fernholz, Ichiba, Karatzas & Prokaj ('13).

- Positive recurrence property

  Pal & Pitman ('10), Dembo & Tsai ('17), ...
- Pathwise differentiability Lipshutz & Ramanan ('19ab).

Still in non-degenerate case.

If a concavity relation of diffusion coefficients

$$\sigma_k^2 \geq \frac{1}{2}(\sigma_{k-1}^2 + \sigma_{k+1}^2)\,, \qquad k = 2, \ldots, n-1$$

holds and the initial value x is away from the triple points, i.e.,

$$\mathrm{x} \notin \{x \in \mathbb{R}^n : x_i = x_j = x_k \text{ for some } i \neq j, j \neq k, k \neq i\}$$

then

$$\mathbb{P}(\tau < \infty) = 0\,,$$

and hence, it is strongly solvable over the time interval $[0, \infty)$.

Ichiba, Karatzas & Shkolnikov ('13)

Ichiba & Sarantsev ('17)

Degenerate cases:

In this talk, we shall consider *degenerate cases* by allowing some of σ_k to be zero in

$$dX_i(t) = \sum_{k=1}^{n} g_k \cdot 1_{\{X_i(t) = R_k(t)\}} dt + \sum_{k=1}^{n} \sigma_k \cdot 1_{\{X_i(t) = R_k(t)\}} dB_i(t)$$

for $i = 1, \ldots, n$, $t \geq 0$.

- For example, $n = 2$:

FERNHOLZ, ICHIBA, KARATZAS & PROKAJ ('13).
(cf. ICHIBA, KARATZAS & PROKAJ ('13),
ICHIBA, KARATZAS, PROKAJ & YAN ('18))

- When $n = 3$, we consider two extreme cases

(i) $\sigma_1 = \sigma_3 = 0$, $\sigma_2 = 1$; (ii) $\sigma_1 = \sigma_3 = 1$, $\sigma_2 = 0$.

Assume the initial value is fixed and away from the triple points

$$\mathrm{x} \notin \{x \in \mathbb{R}^n : x_i = x_j = x_k \text{ for some } i \neq j, j \neq k, k \neq i\}.$$

Proposition ($n = 3$).

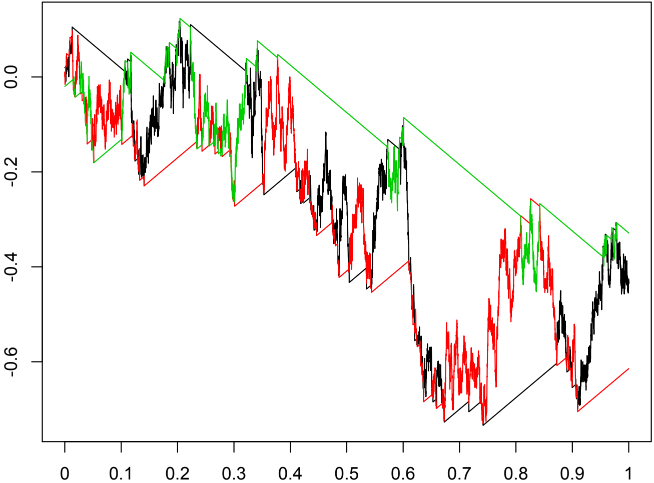
In case (i) $\sigma_1 = \sigma_3 = 0$, $\sigma_2 = 1$ with $n = 3$, the system of equations for $i = 1, 2, 3$

$$X_i(\cdot) = x_i + \sum_{k=1}^{3} \int_0^{\cdot} g_k 1_{\{X_i(t) = R_k(t)\}} \mathrm{d}t + \int_0^{\cdot} 1_{\{X_i(t) = R_2(t)\}} \mathrm{d}B_i(t)$$

admits a pathwise unique, strong solution with the non-sticky condition and there is no triple collision

$$\mathbb{P}(\tau < \infty) = 0 .$$

Simulation: case (i)



Black $= X_1(\cdot)$, Red $= X_2(\cdot)$, Green $= X_3(\cdot)$.

Here we have taken $-g_1 = 1 = g_3$.

Case (ii)

- In case (ii) $\sigma_1 = \sigma_3 = 1$, $\sigma_2 = 0$ with $n = 3$, for the system of equations, $i = 1, 2, 3$

$$X_i(\cdot) = x_i + \sum_{k=1}^{3} \int_0^{\cdot} g_k \, 1_{\{X_i(t) = R_k(t)\}} \, \mathrm{d}t$$

$$+ \int_0^{\cdot} \Big(1_{\{X_i(t) = R_1(t)\}} + 1_{\{X_i(t) = R_3(t)\}} \Big) \mathrm{d}B_i(t) \,,$$

there exists a weak solution, unique in the sense of distribution and

$$L(\cdot; R_1 - R_3) \equiv 0 \,,$$

Case (ii) $\sigma_1 = \sigma_3 = 1, \ \sigma_2 = 0$

$$L(\cdot; R_1 - R_3) \equiv 0 \,,$$

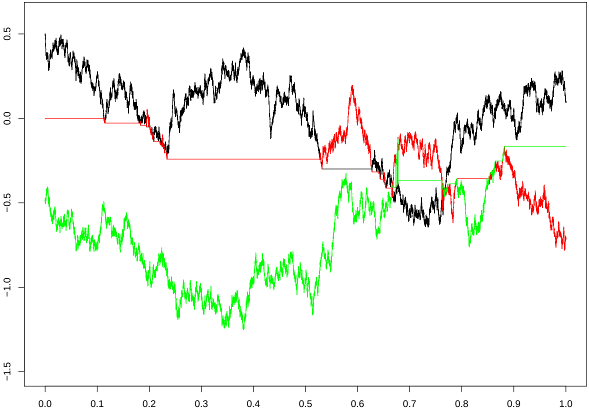
where $L(t; \Xi)$ is the semimartingale local time for a real-valued semimartingale $\Xi(\cdot)$ accumulated at the origin over the time interval $[0, t]$ for $t \geq 0$, i.e.,

$$L(\cdot; \Xi) := L^{\Xi}(\cdot) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^{\cdot} 1_{\{0 \leq \Xi(s) < \varepsilon\}} \langle \Xi \rangle (s) \,.$$

- This solution is path-wise unique and strong *until the first time* τ when the triple collision occurs.
- However, the solution *fails to be strong* after τ. In particular, if $g_1 \leq g_3$, $\mathbb{P}(\tau < \infty) = 1$.

Simulation: case (ii)

$X_1(\cdot)$ (Black) $X_2(\cdot)$ (Red) $X_3(\cdot)$ (Green)
with $g_1 := -0.5$, $g_2 := 0$, $g_3 := 0.5$, $\sigma_1 = \sigma_3 = 1$, $\sigma_2 = 0$.



Analysis of case (i)

$$\sigma_1 = \sigma_3 = 0\,, \quad \sigma_2 = 1\,.$$

Suppose a solution exists. Then the ranked process satisfies

$$R_1^X(t) = x_1 + g_1\, t + \frac{1}{2}\, \Lambda^{(1,2)}(t)\,,$$

$$R_2^X(t) = x_2 + g_2\, t + W(t) - \frac{1}{2}\, \Lambda^{(1,2)}(t) + \frac{1}{2}\, \Lambda^{(2,3)}(t)\,,$$

$$R_3^X(t) = x_3 + g_3\, t - \frac{1}{2}\, \Lambda^{(2,3)}(t)\,,$$

thanks to Banner & Ghomrasni ('08), where

$$W(\cdot) = \sum_{i=1}^{3} \int_0^{\cdot} 1_{\{X_i(t) = R_2^X(t)\}}\, \mathrm{d}B_i(t)$$

is standard Brownian motion by the P. Lévy theorem and we denote by $\Lambda^{(k,\ell)}(t) \equiv L(t; R_k^X - R_\ell^X)$ the local time accumulated at the origin by the continuous, nonnegative semimartingale $R_k^X(\cdot) - R_\ell^X(\cdot)$ over the time interval $[0, t]$.

Gaps $(G(\cdot), H(\cdot))$

We set now

$$G(\cdot) := R_1^X(\cdot) - R_2^X(\cdot), \qquad H(\cdot) := R_2^X(\cdot) - R_3^X(\cdot)$$

for the sizes of the gaps between the leader and the middle particle, and between the middle particle and the laggard, respectively, and obtain the semimartingale representations

$$G(t) = x_1 - x_2 - (g_2 - g_1)\,t - W(t) - \frac{1}{2}\,L^H(t) + L^G(t),$$

$$H(t) = x_2 - x_3 - (g_3 - g_2)\,t + W(t) - \frac{1}{2}\,L^G(t) + L^H(t)$$

for $t \geq 0$.

The gaps are the reflecting Brownian motion in the theory of Harrison & Reiman ('81) but with degeneracy.

Construction of gaps: case (i)

Informed with this analysis, we start with given real numbers g_i , $i = 1, 2, 3$ and $x_1 > x_2 > x_3$, and construct a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ under which $W(\cdot)$ is a standard Brownian motion.

Solving the Skorokhod reflection equation

$$A(t) := \max_{0 \le s \le t} \Big(-(x_1 - x_2) + (g_2 - g_1)\, s + W(s) + \frac{1}{2}\, \Gamma(s) \Big)^{+} ,$$

$$\Gamma(t) := \max_{0 \le s \le t} \Big(-(x_2 - x_3) + (g_3 - g_2)\, s - W(s) + \frac{1}{2}\, A(s) \Big)^{+}$$

we define the gaps by the Skorokhod map

$$G(\cdot) := U(\cdot) + \max_{0 \le s \le \cdot} (-U(s))^{+} , \qquad H(\cdot) := V(\cdot) + \max_{0 \le s \le \cdot} (-V(s))^{+} ,$$

$$U(t) := x_1 - x_2 - (g_2 - g_1)\, t - W(t) - \frac{1}{2}\, \Gamma(t) ,$$

$$V(t) := x_2 - x_3 - (g_3 - g_2)\, t + W(t) - \frac{1}{2}\, A(t)$$

for $t \ge 0$.

Construction of ranks: case (i)

We introduce now

$$R_1(t) := x_1 + g_1\, t + \frac{1}{2}\, A(t)$$

$$R_2(t) := x_2 + g_2\, t + W(t) - \frac{1}{2}\, A(t) + \frac{1}{2}\, \Gamma(t)$$

$$R_3(t) := x_3 + g_3\, t - \frac{1}{2}\, \Gamma(t)$$

for $0 \le t < \infty$ and note the relations

$$R_1(\cdot) - R_2(\cdot) = G(\cdot) \ge 0\,, \qquad R_2(\cdot) - R_3(\cdot) = H(\cdot) \ge 0$$

By Payley-Wiener-Zygmund theorem for $W(\cdot)$

$$\mathbb{P}(R_1(\cdot) - R_3(\cdot) = G(\cdot) + H(\cdot) > 0) = 1\,.$$

"Two ballistic motions cannot squeeze a diffusive (Brownian) motion".

Construction of individual motions: case (i)

We start at time $\tau_0 \equiv 0$ and
follow the paths of the top particle and of the pair consisting of the bottom two particles *separately*, until the top particle collides with the leader of the bottom pair (at time ϱ_0).

Then we follow the paths of the bottom particle and of the pair consisting of the top two particles *separately*, until the bottom particle collides with the laggard of the top pair (at time τ_1).
We repeat the procedure until $\mathcal{S} := \lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \rho_k$

$$0 = \tau_0 \le \varrho_0 \le \tau_1 \le \varrho_1 \le \cdots \le \tau_k \le \varrho_k \le \cdots ,$$

- During each interval of the form $[\tau_k, \varrho_k)$ or $[\varrho_k, \tau_{k+1})$, a pathwise unique, strong solution of the corresponding two-particle system Fernholz et al.('13).

(cf. Karatzas, Pal & Shkolnikov ('12))

Positive recurrence: case (i)

Proposition If the stability condition

$$2(g_3 - g_2) + (g_1 - g_2)^- > 0\,, \quad 2(g_2 - g_1) + (g_2 - g_3)^- > 0$$

holds, then the gap process $(G(\cdot), H(\cdot))$ is positive recurrent, has a unique invariant probability measure $\boldsymbol{\pi}$ with $\boldsymbol{\pi}((0,\infty)^2) = 1$, and converges to this measure in distribution as $t \to \infty$ with the strong laws of large numbers

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(G(t), H(t))\mathrm{d}t = \int_{[0,\infty)^2} f(x,y)\boldsymbol{\pi}(\mathrm{d}x, \mathrm{d}y) \quad a.s.\,;$$

Particularly,

$$\lim_{t\to\infty} \frac{L^G(t)}{t} = \frac{2}{3}(g_2+g_3-2g_1)\,, \quad \lim_{t\to\infty} \frac{L^H(t)}{t} = \frac{2}{3}(2g_3-g_1-g_2)\,.$$

Proof is an extension of Hobson & Rogers ('93).

In the symmetric case $g_2 - g_1 = g_3 - g_2 =: \lambda/2 > 0$, we have

$$\mathbb{E}_\pi[G(t)] = \mathbb{E}_\pi[H(t)] = \frac{1}{3\lambda},$$

and the Laplace transform

$$\widehat{\pi}(a_1, a_2) := \mathbb{E}_\pi[\exp(-a_1 G(t) - a_2 H(t))]; \quad (a_1, a_2) \in [0, \infty)^2 \setminus \{0\},$$

of the joint stationary distribution π satisfies

$$\widehat{\pi}(a_1, a_2) = \frac{\lambda[(2a_1 - a_2)\widehat{\pi}(a_2, a_2) + (2a_2 - a_1)\widehat{\pi}(a_1, a_1)]}{(a_1 - a_2)^2 + \lambda(a_1 + a_2)}.$$

The joint distribution of the gaps is determined by the distribution of the sum under the stationary measure.

- This form also rules out exponential marginal distributions for gaps (cf. HARRISON & WILLIAMS ('87)).
- Ongoing joint work with RASCHEL & FRANCESCHI.

Now let us look at Case (ii)

$$\sigma_1 = \sigma_3 = 1, \; \sigma_2 = 0 \text{ with } n = 3\,, \; i = 1, 2, 3$$

$$X_i(\cdot) = x_i + \sum_{k=1}^{3} \int_0^{\cdot} g_k \, 1_{\{X_i(t) = R_k(t)\}} \, \mathrm{d}t$$

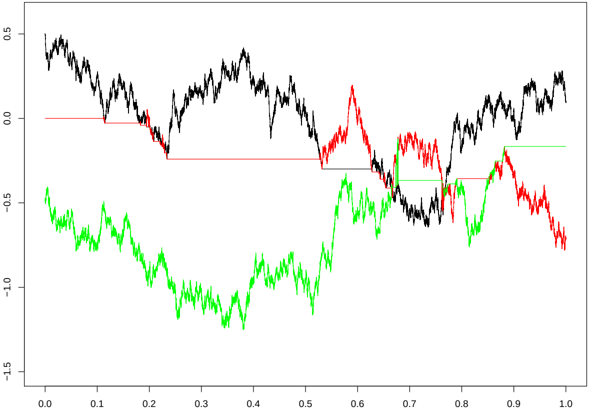
$$+ \int_0^{\cdot} \Big(1_{\{X_i(t) = R_1(t)\}} + 1_{\{X_i(t) = R_3(t)\}} \Big) \mathrm{d}B_i(t)$$

with the no-stickiness condition and no local time from triple collision

$$\int_0^{\infty} 1_{\{R_k(t) = R_\ell(t)\}} \mathrm{d}t = 0\,, \qquad L(\cdot\,; R_1 - R_3) \equiv 0\,,$$

Simulation: case (ii)

$X_1(\cdot)$ (Black) $X_2(\cdot)$ (Red) $X_3(\cdot)$ (Green)
with $g_1 := -0.5$, $g_2 := 0$, $g_3 := 0.5$, $\sigma_1 = \sigma_3 = 1$, $\sigma_2 = 0$.



Analysis: case (ii)

Assume that a solution exists. Then

$$R_1^X(t) = x_1 + g_1\, t + W_1(t) + \frac{1}{2}\,\Lambda^{(1,2)}(t)\,,$$
$$R_2^X(t) = x_2 + g_2\, t - \frac{1}{2}\,\Lambda^{(1,2)}(t) + \frac{1}{2}\,\Lambda^{(2,3)}(t)\,,$$
$$R_3^X(t) = x_3 + g_3\, t + W_3(t) - \frac{1}{2}\,\Lambda^{(2,3)}(t)\,; \quad t \geq 0\,,$$

where $(W_1(\cdot), W_3(\cdot))$ is a standard, two-dimensional Brownian motion:

$$W_k(\cdot) = \sum_{i=1}^{3} \int_0^{\cdot} 1_{\{X_i(t)=R_k^X(t)\}}\, \mathrm{d}B_i(t)\,, \qquad k = 1, 3\,.$$

Then, the gaps $(G(\cdot) =: \varrho_{\cdot} \cos(\vartheta_{\cdot})\,, H(\cdot) =: \varrho_{\cdot} \sin(\vartheta_{\cdot}))$ form a reflecting Brownian motion on a nonnegative quadrant studied by Varadhan & Williams ('85), Williams ('87).

Let us consider the special case $g_1 - g_2 = g_2 - g_3$.
In this case, $(G(\cdot) =: \varrho_. \cos(\vartheta_.), H(\cdot) =: \varrho_. \sin(\vartheta_.))$ is a planar Brownian motion reflected on the faces of the nonnegative quadrant at angles $\theta_1 = \theta_2 = \arctan(1/2)$ relative to the interior normals there, thus with

$$\varphi(\rho, \theta) := \rho^\alpha \cos(\alpha\theta - \theta_1); \quad 0 \leq \rho < \infty, \ 0 \leq \theta \leq \pi/2,$$

$$\alpha := \frac{2}{\pi} \arctan(4/3) \in \left(\frac{1}{2}, \frac{2}{3}\right),$$

the process $\varphi(\varrho_., \vartheta_.)$ is a nonnegative, continuous local submartingale; The process $(G(\cdot), H(\cdot))$, started in the interior, hits eventually the corner a.s. , that is,

$$\mathbb{P}(\tau < \infty) = 1.$$

The two diffusive motions can squeeze the ballistic motion in the middle.

In the Doob-Meyer decomposition of $\varphi(\varrho_\cdot, \vartheta_\cdot)$, the continuous, adapted, non-decreasing process is a constant multiple of the positive, continuous, additive functional $\Lambda^\bullet(\cdot) :=$

$$\frac{\alpha(2-\alpha)}{2} \lim_{\varepsilon \downarrow 0} \varepsilon^{1-(2/\alpha)} \int_0^\cdot (\cos(\alpha\vartheta_t - \theta_1))^{(2/\alpha)-2} \cdot 1_{[0,\varepsilon)}(\varphi(\varrho_t, \vartheta_t))\mathrm{d}t\,,$$

where the limit is in the sense of convergence in probability.

- If $G(0) = H(0) = 0$, the right continuous inverse of $t \mapsto \Lambda^\bullet(t)$ is a stable subordinator of index $\alpha/2$.
- The set of triple collision

$$\{t : G(t) = H(t) = 0\}$$

has HAUSSDORFF dimension $\alpha/2$.

(cf. WILLIAMS ('87), ROGERS ('89) for details.)

- In particular, $L^\varrho(\cdot) \equiv 0$ and $L^{G+H}(\cdot) \equiv 0$.

Construction of ranks and gaps: case (ii)

We introduce the ranks: for $t \geq 0$

$$R_1(t) := x_1 + g_1\, t + W_1(t) + \frac{1}{2} A(t)\,,$$

$$R_2(t) := x_2 + g_2\, t - \frac{1}{2} A(t) + \frac{1}{2} \Gamma(t)\,,$$

$$R_3(t) := x_3 + g_3\, t + W_3(t) - \frac{1}{2} \Gamma(t)\,,$$

$$A(t) = \max_{0 \leq s \leq t} \Big(-(x_1 - x_2) + (g_2 - g_1)\, s - W_1(s) + \frac{1}{2} \Gamma(s) \Big)^+\,,$$

$$\Gamma(t) = \max_{0 \leq s \leq t} \Big(-(x_2 - x_3) + (g_3 - g_2)\, s + W_3(s) + \frac{1}{2} A(s) \Big)^+\,,$$

$$U(t) := x_1 - x_2 - (g_2 - g_1)\, t + W_1(t) - \frac{1}{2} \Gamma(t)\,,$$

$$V(t) := x_2 - x_3 - (g_3 - g_2)\, t - W_3(t) - \frac{1}{2} A(t)\,,$$

and $(G(\cdot), H(\cdot))$ as the Skorokhod map of $(U(\cdot), V(\cdot))$.

Construction of individual motions: Case (ii)

We mimic a construction of WALSH Brownian motion:

PROKAJ ('09), ICHIBA, KARATZAS, PROKAJ & YAN ('18).

We define the first passage time and the zero sets :

$$\sigma_0 := \inf\{t > 0 : G(t) \wedge H(t) = 0\},$$

$$\{t \geq 0 : G(t) = 0\}, \quad \{t \geq 0 : H(t) = 0\},$$

and the corresponding countable excursion intervals $\{\mathcal{C}_\ell^G, \ell \in \mathbb{N}\}$, $\{\mathcal{C}_m^H, m \in \mathbb{N}\}$, i.e.,

$$\mathbb{R}_+ \backslash \{t : G(t) = 0\} = \bigcup_{\ell \in \mathbb{N}} \mathcal{C}_\ell^G, \quad \mathbb{R}_+ \backslash \{t : H(t) = 0\} = \bigcup_{\ell \in \mathbb{N}} \mathcal{C}_\ell^H .$$

Then, by enlarging the probability space, for each excursion after σ_0, we *assign random permutation matrices* to interchange the indexes with probability $1/2$ and not to do so with probability $1/2$;

Then apply Theorem VI.1.10 of REVUZ & YOR ('99).

Uniqueness

- Up to the first time τ of triple collision, the solution is pathwise unique and strong, and hence, the distribution is uniquely determined. After τ, because of invariant under permutations, each $X_i(t)$ is $R_k(t)$, $k = 1, 2, 3$ equally likely with probability $1/3$. By the Markov property of $R_k(\cdot)$, the weak solution is unique in distribution.

- The permutation at the excursion starting from the corner does not contribute to the distribution but *perturb the pathwise behavior*. Thus, the pathwise uniqueness fails, but then by the dual of Yamada-Watanabe theorem (cf. Cherny ('01), Engelbert ('91)), it is not strong after the first triple collision time τ.

- Open problem: determine stochastic flows as in Tsirel'son ('97), Warren ('02), Le Jan& Raimond ('04), Watanabe ('00), Akahori, Izumi & Watanabe ('09).

Summary:

Degenerate, rank-based particle systems

- $N = 2$ Fernholz, Ichiba, Karatzas & Prokaj ('13).
- Two extreme cases: $N = 3$
  - (i) $\sigma_1 = 0 = \sigma_3\,,\ \sigma_2 = 1\,;$
  - (ii) $\sigma_2 = 0\,,\ \sigma_1 = 1 = \sigma_3\,.$
- Skew-elastic collision
- $N \geq 4$:
  - multiple collisions
    (cf. Ichiba & Sarantsev ('17) for non-degenerate case)
  - Extreme cases:
    Case (i) $\sigma_1 = 0\,,\ \sigma_2 = \cdots = \sigma_{n-1} = 1\,,\ \sigma_n = 0\,,$
    Case (ii) $\sigma_{2i} = 0\,,\ \sigma_{2i-1} = 1\,,\ i = 1, \ldots;$
    Case (iii) $\sigma_{2i} = 1\,,\ \sigma_{2i-1} = 0\,,\ i = 1, \ldots;$

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A system with skew-elastic collisions

Let us modify the first system. For $i = 1, 2, 3$,

$$X_i(\cdot) = x_i + \sum_{k=1}^{3} \int_0^{\cdot} g_k 1_{\{X_i(t) = R_k(t)\}} \mathrm{d}t + \int_0^{\cdot} 1_{\{X_i(t) = R_2(t)\}} \mathrm{d}B_i(t)$$

$$+ \int_0^{\cdot} 1_{\{X_i(t) = R_2(t)\}} \mathrm{d}L^{R_2 - R_3}(t) + + \int_0^{\cdot} 1_{\{X_i(t) = R_3(t)\}} \mathrm{d}L^{R_2 - R_3}(t)$$

with the non-stickiness condition and no local time from the triple collision

$$\int_0^{\infty} 1_{\{R_k(t) = R_\ell(t)\}} \mathrm{d}t = 0 , \ k \neq \ell , \quad L^{R_1 - R_3}(\cdot) \equiv 0 .$$

From ranks $(R_1(\cdot), R_2(\cdot), R_3(\cdot))$ to $(X_1(\cdot), X_2(\cdot), X_3(\cdot))$

From the analysis of ranks,

$$R_1(t) \;=\; x_1 + g_1\, t + \frac{1}{2}\,\Lambda^{(1,2)}(t)$$

$$R_2(t) \;=\; x_2 + g_2\, t + W(t) - \frac{1}{2}\,\Lambda^{(1,2)}(t) + \frac{3}{2}\,\Lambda^{(2,3)}(t)$$

$$R_3(t) \;=\; x_3 + g_3\, t + \frac{1}{2}\,\Lambda^{(2,3)}(t)$$

the solution can be constructed similarly as before.

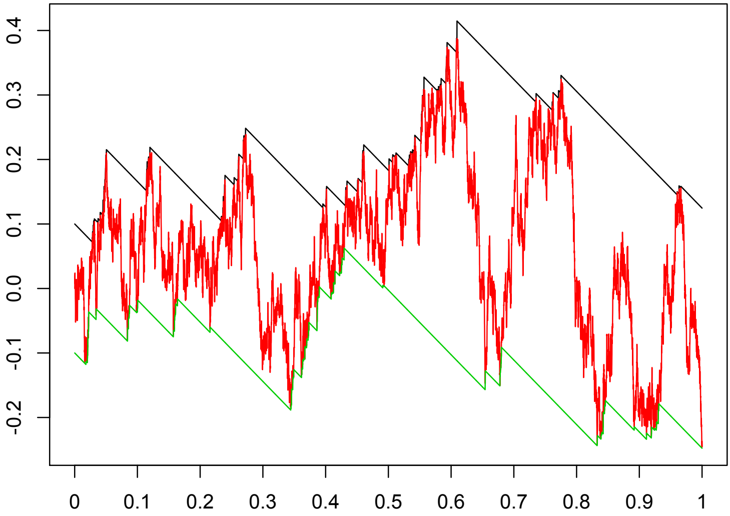
- The gap process $(G(\cdot), H(\cdot))$ is a two-dimensional, degenerate, reflecting Brownian motion with oblique reflection.

Proposition. The degenerate system with skew-elastic collisions:

$$X_i(\cdot) = x_i + \sum_{k=1}^{3} \int_0^{\cdot} g_k 1_{\{X_i(t) = R_k(t)\}} \mathrm{d}t + \int_0^{\cdot} 1_{\{X_i(t) = R_2(t)\}} \mathrm{d}B_i(t)$$

$$+ \int_0^{\cdot} 1_{\{X_i(t) = R_2(t)\}} \mathrm{d}L^{R_2 - R_3}(t) + \int_0^{\cdot} 1_{\{X_i(t) = R_3(t)\}} \mathrm{d}L^{R_2 - R_3}(t)$$

has a pathwise-unique, strong solution with non-stickiness condition until the time of triple collisions.



Black $= R_1(\cdot)$, Red $= R_2(\cdot)$, Green $= R_3(\cdot)$. Here we have taken $g_1 = -1$, $g_2 = -2$ and $g_3 = -1$.

Invariant distribution of gaps

Under the conditions

$$3\,g_3 > 2\,g_1 + g_2\,, \qquad 2\,g_3 > g_1 + g_2\,,$$

the gap process $(G(\cdot), H(\cdot))$ is positive recurrent and has a unique invariant probability measure

$$\pi(\mathrm{d}x_1, \mathrm{d}x_2) = 4\lambda_1\lambda_2 \exp(-2\lambda_1 x_1 - 2\lambda_2 x_2)\,, (x_1, x_2) \in (0, \infty)^2\,,$$

where $\lambda_1 := 2(3\,g_3 - 2\,g_1 - g_2)\,,\ \lambda_2 := 2(2\,g_3 - g_1 - g_2)\,.$
Indeed, by an application of ITÔ's formula,

$$\mathrm{d}\big(G^2(t) + 3G(t)H(t) + 3H^2(t)\big)$$

$$= \Big[1 - \frac{1}{2}\big(\lambda_1 G(t) + 3\lambda_2 H(t)\big)\Big]\mathrm{d}t + \big(3H(t) + G(t)\big)\,\mathrm{d}W(t)\,; \quad t \geq 0\,.$$

suggests the function $V(g,h) = \exp\{\sqrt{g^2 + 3gh + 3h^2}\}$ is a LYAPOUNOV function for the semimaritngale reflecting Brownian motion.

cf. DAI & KURTZ ('03) O'CONNELL & ORTMANN ('12)