Extended Lévy's Theorem for a Two-Sided Reflection and Applications to the Stroock-Williams Problem

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40 years of reflected Brownian motion and related topics

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Abstract

We state an extension of Extended Lévy's theorem.

This theorem gives an explicit realisation of a reflecting Brownian motion with drift $-\mu$, started at x, reflecting above zero and below one, and its local time at zero.

We then present a solution to an initial boundary value problem of Stroock and Williams with sticky boundary conditions at zero and one.

Extended Lévy's theorem

Theorem (Extended Lévy's theorem)

For $B_t^\mu = B_t + \mu t$*, and* $S_t^\mu = S_{0,t}^\mu := \sup_{0\le u\le t} B_u^\mu$*,* $0 \le s \le t$*, we have the following explicit realisation for a reflecting Brownian motion* $R_{0+}^{-\mu,x}$ *with drift* $-\mu$*, started at* $x \ge 0$*, and reflected above zero*

$$Y^x = (x \vee S^\mu) - B^\mu = ((x \vee S_t^\mu) - B_t^\mu)_{t\ge 0}. \quad (1)$$

Furthermore, we have the following identity in law

$$((x \vee S^\mu) - B^\mu, (x \vee S^\mu) - x) \overset{law}{=} (R_{0+}^{-\mu,x}, \ell^0(R_{0+}^{-\mu,x})) \quad (2)$$

where $\ell^0(R_{0+}^{-\mu,x})$ *is the local time of* $R_{0+}^{-\mu,x}$ *at* 0*.*

Skorokhod Map

Result (1.8) in paper [1] (Kruk et al 2008) tells us that for the space of right-continuous functions with left limits taking values in $\mathbb{R}$, the double Skorokhod map $\Gamma_{0,a}$ on $[0, a]$ has the explicit realisation

$$\Gamma_{0,a}(\psi)(t) = \psi(t) - \left[(\psi(0) - a)^{+} \wedge \inf_{u \in [0,t]} \psi(u)\right] \vee \sup_{s \in [0,t]} \left[(\psi(s) - a) \wedge \inf_{u \in [s,t]} \psi(u)\right] \quad (3)$$

for $t \geq 0$ and $a > 0$.

Skorokhod Map

From (3), let $\psi(t) = x - B_t^\mu$ where $B_t^\mu = B_t + \mu t$ and $B = (B_t)_{t \geq 0}$ is a standard Brownian motion started at zero, $\mu \in \mathbb{R}$ is a given and fixed constant, and take $a = 1$. This yields a process Z^x reflecting between 0 and 1 given by

$$Z_t^x := -B_t^\mu + (x \vee S_{0,t}^\mu) \wedge \inf_{s \in [0,t]} [(1 + B_s^\mu) \vee S_{s,t}^\mu]. \tag{4}$$

Visualisation of Z^x

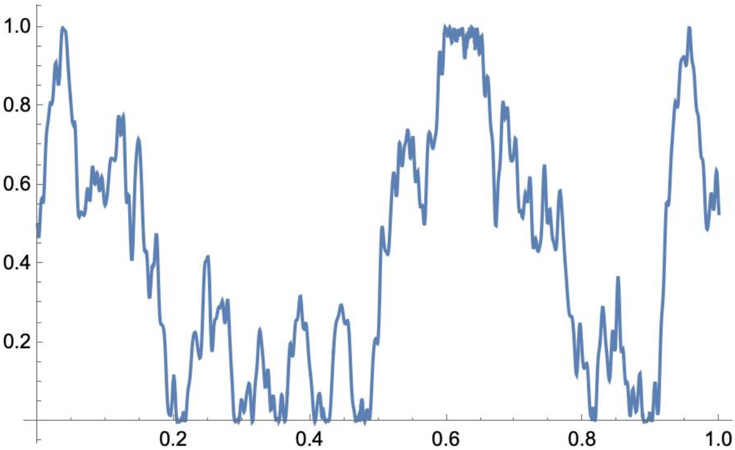

Figure: A visualisation in Mathematica of the realisation Z^x of the reflecting Brownian motion for $x = 0.5$.

Geometric Intuition

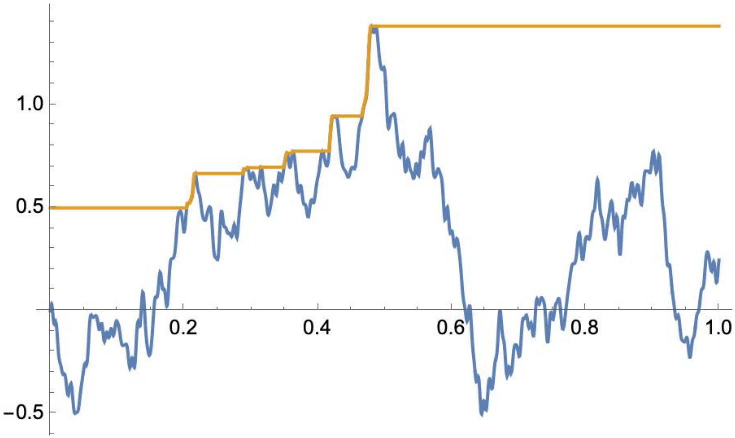

Figure: The blue line shows B^{μ}, whilst the orange line depicts $(x \vee S^{\mu})$ for $x = 0.5$

Geometric Intuition

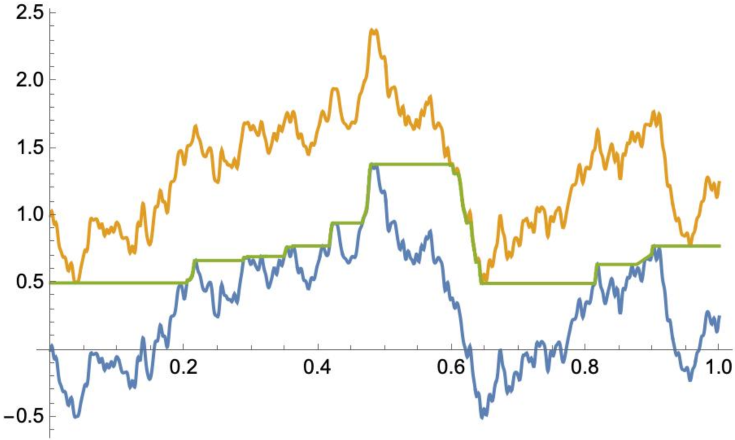

Figure: The blue line show the Brownian motion B^{μ}, the orange line shows $1 + B^{\mu}$, whilst the green line depicts $Q_t^x = \left(x \vee S_{0,t}^{\mu}\right) \wedge \inf_{s \in [0,t]} \left[(1 + B_s^{\mu}) \vee S_{s,t}^{\mu}\right]$ for $x = 0.5$.

Extended Lévy's theorem for a two-sided reflection

Theorem (Extended Lévy's theorem for a two-sided reflection)

The following identity in law holds

$$\Big((x \vee S^{\mu}_{0,t}) \wedge \inf_{s\in[0,t]}[(1+B^{\mu}_s) \vee S^{\mu}_{s,t}] - B^{\mu}_t, (x \vee S^{\mu}_{0,t}) \wedge \inf_{s\in[0,t]}[(1+B^{\mu}_s) \vee S^{\mu}_{s,t}] - x\Big) \tag{5}$$

$$\overset{law}{=} (R^{-\mu,x}_{0,1}, \ell^0_t(R^{-\mu,x}_{0,1}) - \ell^1_t(R^{-\mu,x}_{0,1}))$$

where $\ell^a(R^{-\mu,x}_{0,1})$ is the symmetric local time of $R^{-\mu,x}_{0,1}$ at a.

Integral form

We can also see that we have the following two integral forms for the local times

$$\ell_t^0(Z^x) = \int_0^t \mathbb{1}_{Z_s^x=0}\, dQ_s^x \tag{6}$$

$$\ell_t^1(Z^x) = -\int_0^t \mathbb{1}_{Z_s^x=1}\, dQ_s^x. \tag{7}$$

A coupled pair of local time realisations

Theorem (A coupled pair of local time realisations)

The following identities in law hold

$$\ell_t^0(R_{0,1}^{-\mu,x}) \overset{law}{=} x \vee \sup_{s\in[0,t]} (B_s^\mu + \ell_s^1(Z^x)) - x \tag{8}$$

$$-\ell_t^1(R_{0,1}^{-\mu,x}) \overset{law}{=} x \wedge \inf_{s\in[0,t]} (B_s^\mu + 1 - \ell_s^0(Z^x)) - x \tag{9}$$

where $\ell^a(R_{0,1}^{-\mu,x})$ is the local time of $R_{0,1}^{-\mu,x}$ at a.

From here we may also see an alternate realisation of Z^x, namely

$$\begin{aligned} Z_t^x &= \ell_t^0(Z^x) - \ell_t^1(Z^x) + x - B_t^\mu \\ &= x \vee \sup_{s\in[0,t]} (B_s^\mu + \ell_s^1(Z^x)) + x \wedge \inf_{s\in[0,t]} (B_s^\mu + 1 - \ell_s^0(Z^x)) - x - B_t^\mu . \end{aligned} \tag{10}$$

Stroock-Williams Problem

Consider the following initial boundary value problem

$$u_t = \mu u_x + \frac{1}{2} u_{xx} \qquad (t > 0, x \geq 0) \qquad (11)$$

$$u(0, x) = f(x) \qquad (x \geq 0) \qquad (12)$$

$$u_t(t, 0) = \nu u_x(t, 0) \qquad (t > 0) \qquad (13)$$

where $\mu, \nu \in \mathbb{R}$.

A probabilistic solution

Theorem (Peskir 2013)

(i) *Let* $f \in C_b^1([0,\infty))$ *and* $f(\infty), f'(\infty) = 0$. *Then there exists a unique* $u(t,x) \in C^\infty((0,\infty) \times [0,\infty))$ *with* $u, u_x \in C_b((0,T] \times [0,\infty))$ *for* $T > 0$ *and* $u(t,\infty) = 0$ *which solves the above initial boundary value problem.*

(ii) *Let* X *be a reflecting Brownian motion starting at* x *under* $\mathbb{P}_x$ *with drift* μ. *The solution* $u(t,x)$ *has a probabilistic representation given by*

$$u(t,x) = \mathbb{E}_x[f(x \vee S_t^{-\mu} - B_t^{-\mu})] - \mathbb{E}_x\left[f'(x \vee S_t^{-\mu} - B_t^{-\mu}) \int_0^{x \vee S_t^{-\mu} - x} e^{-2(\nu-\mu)s} ds\right].$$

Consider the following initial boundary value problem

$$u_t = \mu u_x + \frac{1}{2} u_{xx} \qquad (t > 0, x \geq 0) \qquad (14)$$

$$u(0, x) = f(x) \qquad (x \geq 0) \qquad (15)$$

$$u_t(t, 0) = \nu u_x(t, 0) \qquad (t > 0) \qquad (16)$$

$$u_t(t, 1) = \kappa u_x(t, 1) \qquad (t > 0) \qquad (17)$$

where $\mu, \nu, \kappa \in \mathbb{R}$.

A probabilistic solution

Here is the solution for the case when $\nu = \kappa$

$$u(t,x) = -\mathbb{E}_x\left[\int_{x\vee S_t^{-\mu}-B_t^{-\mu}}^{x\vee S_t^{-\mu}\vee \inf_{s\in[0,t]}[(1+B_s^{-\mu})\vee S_{s,t}^{-\mu}]-B_t^{-\mu}} f'(s)ds\right] \tag{18}$$

$$-\mathbb{E}_x\Bigg[f'((x\vee S_{0,t}^{-\mu})\wedge \inf_{s\in[0,t]}[(1+B_s^{-\mu})\vee S_{s,t}^{-\mu}]-B_t^{-\mu})$$

$$\int_{0\wedge(\inf_{s\in[0,t]}[(1+B_s^{-\mu})\vee S_{s,t}^{-\mu}]-(x\vee S_t^{-\mu}))}^{(x\vee S_{0,t}^{-\mu})\wedge \inf_{s\in[0,t]}[(1+B_s^{-\mu})\vee S_{s,t}^{-\mu}]-x} e^{-2(\nu-\mu)s}ds\Bigg]$$

$$+\mathbb{E}_x\left[\int_{0\vee((x\vee S_t^{-\mu})-\inf_{s\in[0,t]}[(1+B_s^{-\mu})\vee S_{s,t}^{-\mu}])}^{\infty} f'(\inf_{s\in[0,t]}[(1+B_s^{-\mu})\vee S_{s,t}^{-\mu}]-B_t^{-\mu})ds\right]$$

Proof

The proof begins by setting $v = u_x$ and $\lambda = 2(\nu - \mu)$. We then perform a substitution with the original initial boundary value problem for u resulting in the following new initial boundary value problem for v

$$v_t = \mu v_x + \frac{1}{2} v_{xx} \qquad (t > 0, x \geq 0) \qquad (19)$$

$$v(0, x) = f'(x) \qquad (x \geq 0) \qquad (20)$$

$$v_x(t, 0) = \lambda v(t, 0) \qquad (t > 0) \qquad (21)$$

$$v_x(t, 1) = \lambda v(t, 1) \qquad (t > 0) \qquad (22)$$

where we note the boundary conditions have changed from sticky boundary behaviour to elastic boundary behaviour (reflecting for $\lambda = 0$).

Proof

Set

$$Z_t^x = Q_t^{x,-\mu} - B_t^{-\mu} \quad \text{and} \quad \ell_t^0(Z^x) - \ell_t^1(Z^x) = Q_t^{x,-\mu} - x. \tag{23}$$

Let time run backwards and apply Itô's formula to

$$e^{-\lambda(\ell_s^0(Z^x)-\ell_s^1(Z^x))} v(t-s, Z_s^x) \tag{24}$$

to find

$$\begin{aligned} & e^{-\lambda(\ell_s^0-\ell_s^1)} v(t-s, Z_s^x) \\ &= v(t,x) + \int_0^s e^{-\lambda(\ell_r^0-\ell_r^1)} (v_x - \lambda v)(t-r, Z_r^x) dQ_r^{x,-\mu} \\ &\qquad + \int_0^s e^{-\lambda(\ell_r^0-\ell_r^1)} (\mu v_x - v_t + \frac{1}{2} v_{xx})(t-r, Z_r^x) dr \\ &\qquad - \int_0^s e^{-\lambda(\ell_r^0-\ell_r^1)} v_x(t-r, Z_r^x) dB_r \\ &= v(t,x) - M_s \end{aligned} \tag{25}$$

where $M_s = \int_0^s e^{-\lambda(\ell_r^0-\ell_r^1)} v_x(t-r, Z_r^x) dB_r$ is a continuous local martingale for $s \in [0, t)$.

Proof

This gives

$$v(t,x) = e^{-\lambda(\ell_s^0 - \ell_s^1)} v(t-s, Z_s^x) + M_s. \quad (26)$$

From here by choosing an appropriate sequence of stopping times, taking expectation of the above equation, applying the optional sampling theorem and then dominated convergence theorem, we find that

$$v(t,x) = \mathbb{E}[e^{-\lambda(\ell_t^0(Z^x) - \ell_t^1(Z^x))} f'(Z_t^x)] \quad (27)$$

then recalling that $v = u_x$ and $u(t,\infty) = 0$ we find

$$u(t,x) = -\int_x^\infty \mathbb{E}[e^{-\lambda(\ell_t^0(Z^y) - \ell_t^1(Z^y))} f'(Z_t^y)]\, dy \quad (28)$$

and it is at this point that things start to get very algebra heavy.

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The End

Questions? Comments?