An inverse Pitman's theorem for a space-time brownian motion in a type A_1^1 Weyl chamber

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Plan of the presentation

1. Pitman's theorem and transforms
   - One-dimensional Pitman's theorem
   - Pitman's transform and fundamental property
   - Generalized Pitman's transform
2. Generalizations of Pitman's theorem
   - Fundamental domain for the reflection group
   - Generalization of Pitman's theorem in a semisimple Lie algebra
   - A Pitman type theorem in an affine Lie algebra A_1^1
   - An inverse theorem for an affine Lie algebra A_1^1

One-dimensional Pitman's theorem

Pitman's theorem (1975)

Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion then $(B_t - 2\inf_{0\leq s\leq t} B_s)_{t\geq 0}$ is a Bessel process (of dimension 3).

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A **Bessel process of dimension** 3 is a stochastic process $(X_t)_{t\geq 0}$ which is the norm of a three-dimensional Brownian motion $X_t := ||B_t^{(3)}||\ \forall t \geq 0$. Where $B_t^{(3)} := (B_t^1, B_t^2, B_t^3)$ with B_t^1, B_t^2, B_t^3 are three independent one-dimensional Brownian motion.

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With probabilistic tool we can see a Bessel process of dim. 3 like a **Brownian motion conditioned to stay forever positive.**

Definition of Pitman's transform

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Definition (Pitman's transform)

The Pitman's transform denoted by $\mathscr{P}_\alpha$ is defined on the set of continuous paths $\pi : [0, T] \to V$ verifying $\pi(0) = 0$ by the formula :

$$\mathscr{P}_\alpha \pi(t) := \pi(t) - \inf_{0 \le s \le t} \alpha^\vee(\pi(s))\,\alpha$$

Fundamental property of Pitman's transform

Fundamental property

Let η be a path verifying $\eta(0) = 0$ and $\alpha^\vee(\eta(t)) \geq 0\ \forall t \in [0, T]$. Let $\xi \in [0, \alpha^\vee(\eta(t))]$ then it exists a unique path π such as :

$$\begin{cases} \mathscr{P}_\alpha \pi = \eta \\ \xi = -\inf_{0 \leq t \leq T} \alpha^\vee(\pi(t)) \end{cases}$$

Moreover, this path is given by the formula :

$$\pi(t) = \eta(t) - \min\left(\xi, \inf_{t \leq s \leq T} \alpha^\vee\left(\eta(s)\right)\right) \alpha := I_\alpha^\xi \eta(t) \tag{1}$$

(Inversion formula of Pitman due to Biane-Bougerol-O'Connell).

Generalized Pitman's transform

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• In the following we are interested in the Pitman's transform associated to a composition of reflections $w = s_1 \cdots s_q$ which we define by :

$$\mathscr{P}_w := \mathscr{P}_{s_1} \cdots \mathscr{P}_{s_q}$$

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• The composition of these reflections form a group.

• The fundamental domain for this group is called in the vocabulary of Lie algebra a **fundamental Weyl chamber**.

Fundamental domain for the reflection group

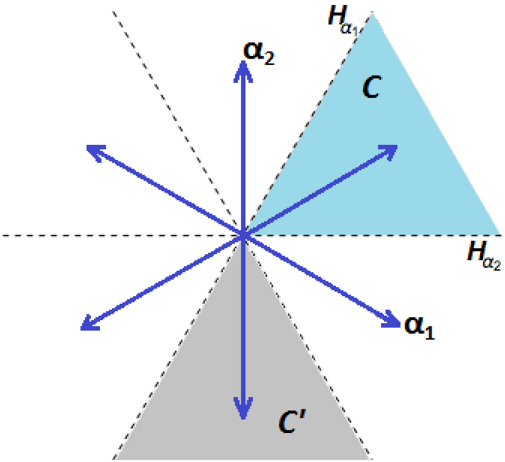

Figure – Fundamental Weyl chamber (in blue) for a basis $\{\alpha_1, \alpha_2\}$.

Generalization of Pitman's theorem in a semisimple Lie algebra

Let V be a semisimple Lie algebra, we can take for example $\mathbb{R}^2$ with a basis of vectors with angle $\frac{\pi}{3}$.
Let w_0 be the longest element of the reflection group.

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Theorem (Biane-Bougerol-O'Connell, 2004)

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We recognize the Pitman's theorem (1975) if we take $V := \mathbb{R}$.
Then the (closed) fundamental Weyl chamber is $\overline{\mathcal{C}} = [0, +\infty[$.
We recognize also $\mathscr{P}_{w_0}$ the initial Pitman's transform.

Illustration of the theorem

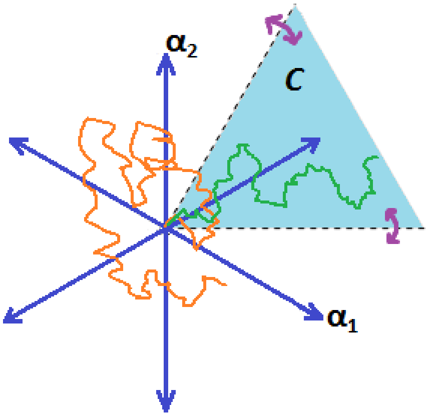

Figure – Illustration of the theorem with a basis $\{\alpha_1, \alpha_2\}$

A Pitman type theorem in an affine Lie algebra A_1^1

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We define a space-time drifted Brownian motion by :

$$B_t^{(\mu)} := (t, B_t + \mu t)$$

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• We define the process $(A_t^{(\mu)})_{t\geq 0}$ like the process $(B_t^{(\mu)})_{t\geq 0}$ conditioned to stay forever in the cone :

$$C := \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid 0 < x < t\}$$

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• These vectors lead to define Pitman transforms acting on a continuous map $\eta : \mathbb{R}_+ \to \mathbb{R}^2$ where $\eta(t) := (t, f(t))$ by :

$$\mathscr{P}_{s_0}\eta(t) := (t, f(t) + 2 \inf_{s \leq t}(s - f(s)))$$

$$\mathscr{P}_{s_1}\eta(t) := (t, f(t) - 2 \inf_{s \leq t} f(s))$$

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• We need to define Lévy transforms by :

$$\mathcal{L}_{s_0}\eta(t) := (t, f(t) + \inf_{s\leq t}(s - f(s)))$$

$$\mathcal{L}_{s_1}\eta(t) := (t, f(t) - \inf_{s\leq t} f(s))$$

• We define $\mathscr{P}_{s_k} := \mathscr{P}_{s_0}$ for k even and $\mathscr{P}_{s_k} := \mathscr{P}_{s_1}$ for k odd.
And similarly for $\mathcal{L}_{s_k}$.

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And similarly for $\mathcal{L}_{s_k}$.

Theorem (Bougerol-Defosseux, 2018)

Let $t \geq 0$, then :

$$\lim_{n \to +\infty} \mathcal{L}_{s_{n+1}} \mathscr{P}_{s_n} \cdots \mathscr{P}_{s_1} \mathscr{P}_{s_0} B^{(\mu)} \quad \text{exists a.s.}$$

and the limit process has the same distribution as $A^{(\mu)}$.

An inverse theorem for an affine Lie algebra A_1^1

• We define two transforms acting on a continuous map $\eta : \mathbb{R}_+ \to \mathbb{R}^2$ where $\eta(t) := (t, f(t))$ by : $I_0^\xi \eta(t) := (t, f(t) + 2\min(\xi, \inf_{s\geq t}(s - f(s))))$ and $I_1^\xi \eta(t) := (t, f(t) - 2\min(\xi, \inf_{s\geq t}(s)))$.

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- Let $I_k^\xi := I_0^\xi$ for k even and $I_k^\xi := I_1^\xi$ for k odd.

Theorem (Defosseux-Hérent, 2022)

It exists a double sequence of random variables $(\xi_{k,p})_{k,p}$ such that the sequence of processes :

$$\left\{ I_0^{\xi_{0,p}} \cdots I_p^{\xi_{p,p}} A^{(\mu)}(t), \ t \geq 0 \right\}_{p \geq 0}$$

converges, in the sense of finite dimensional distributions, towards the space-time Brownian motion $\{B_t, \ t \geq 0\}$.

Thank you for your attention