Lévy driven non-linear Langevin type equations

Mihai Gradinaru

University of Rennes, France - Institut de Recherche Mathématique de Rennes

40 years of Reflected Brownian Motion and related topics
Roscoff, April 24-28th 2023

Folklore : original Langenvin equation in fluid dynamics. It concerns the motion of a particle with a position x^σ subject to a friction force $-\dot{x}^\sigma$ and a white noise $\dot{b}$ with amplitude σ

$$\ddot{x}^\sigma = -\dot{x}^\sigma + \sigma \dot{b}.$$

Denoting $v^\sigma = \dot{x}^\sigma$ we can write

$$v_t^\sigma = v_0 - \int_0^t f(v_s^\sigma)\mathrm{d}s + \sigma b_t \text{ and } x_t^\sigma = x_0 + \int_0^t v_s^\sigma \mathrm{d}s$$

with b Brownian motion and drag force $f(v) = v$
(integrated Ornstein-Uhlenbeck process).

In several physical situations the viscous drag force could be non-linear $f(v) = v^2$ (Rayleigh quadratic drag) or $f(v) = v^3$ (Stokes aerodynamics), but also in a model from the biology (the motion of Kuhlia Mugli fish) (also called persistent turning walker model having a diffusive behaviour in large time): first extension: polynomial function (outline questions, result and, if time, some ideas of proof)

$$f(v) = \mathrm{sgn}(v)|v|^\beta, \ \beta \geq 1.$$

Some models provide a drag force which depends on time (for instance for a rocket with respect to the atmospheric friction, ...) first extension continued: a time-inhomogeneous drift (focus on).

$$f(t, v) = t^{-\gamma}\mathrm{sgn}(v)|v|^\beta, \ \beta \geq 1, \gamma \in \mathbb{R}, t > 0.$$

Another further extension ongoing work (possibly interesting) about random (and possibly time-inhomogeneous) environment extending the so-called Brox process: if $W(v)$ is a Brownian motion indexed by $v \in \mathbb{R}$ we take

$$f(t, v) = t^{-\gamma}\dot{W}(v), \ \gamma \in \mathbb{R}, t > 0.$$

In several physical situations the viscous drag force could be non-linear $f(v) = v^2$ (Rayleigh quadratic drag) or $f(v) = v^3$ (Stokes aerodynamics), but also in a model from the biology (the motion of Kuhlia Mugli fish) (also called persistent turning walker model having a diffusive behaviour in large time): first extension: polynomial function (outline questions, result and, if time, some ideas of proof)

$$f(v) = \mathrm{sgn}(v)|v|^\beta, \ \beta \geq 1.$$

Some models provide a drag force which depends on time (for instance for a rocket with respect to the atmospheric friction, ...) first extension continued: a time-inhomogeneous drift (focus on).

$$f(t, v) = t^{-\gamma}\mathrm{sgn}(v)|v|^\beta, \ \beta \geq 1, \gamma \in \mathbb{R}, \ t > 0.$$

Another further extension ongoing work (possibly interesting) about random (and possibly time-inhomogeneous) environment extending the so-called Brox process: if $W(v)$ is a Brownian motion indexed by $v \in \mathbb{R}$ we take

$$f(t, v) = t^{-\gamma}\dot{W}(v), \ \gamma \in \mathbb{R}, \ t > 0.$$

In several physical situations the viscous drag force could be non-linear $f(v) = v^2$ (Rayleigh quadratic drag) or $f(v) = v^3$ (Stokes aerodynamics), but also in a model from the biology (the motion of Kuhlia Mugli fish) (also called persistent turning walker model having a diffusive behaviour in large time): first extension: polynomial function (outline questions, result and, if time, some ideas of proof)

$$f(v) = \mathrm{sgn}(v)|v|^\beta, \ \beta \geq 1.$$

Some models provide a drag force which depends on time (for instance for a rocket with respect to the atmospheric friction, ...) first extension continued: a time-inhomogeneous drift (focus on).

$$f(t, v) = t^{-\gamma}\mathrm{sgn}(v)|v|^\beta, \ \beta \geq 1, \gamma \in \mathbb{R}, \ t > 0.$$

Another further extension ongoing work (possibly interesting) about random (and possibly time-inhomogeneous) environment extending the so-called Brox process: if $W(v)$ is a Brownian motion indexed by $v \in \mathbb{R}$ we take

$$f(t, v) = t^{-\gamma}\dot{W}(v), \ \gamma \in \mathbb{R}, \ t > 0.$$

Recent work concerning systems in physics, chemistry, biology, economics or finance, brings the interest for a non-Gaussian noise and in particular for jump Lévy noise.

Examples : spectral lines broadening in plasmas, or systems with long-range interactions, or behavioural pattern of albatrosses or sharks (Lévy flights), or some climate models. So:

second extension : the considered noise will be ℓ_t, a symmetric α-stable Lévy process which is a pure jump with càdlàg paths process.

Keep in mind the following main example

$$v_t^\sigma = -\int_{t_0}^t \frac{\operatorname{sign}(v_s^\sigma)|v_s^\sigma|^\beta}{s^\gamma} \mathrm{d}s + \sigma \ell_t \quad \text{and} \quad x_t^\sigma = \int_{t_0}^t v_s^\sigma \mathrm{d}s$$

with ℓ symmetric α-stable Lévy process, $\alpha \in (0, 2]$, $\beta \geq 1$, $\gamma \in \mathbb{R}$, initial conditions v_{t_0}, x_{t_0}. Note that $\alpha = 2$ corresponds to a Brownian driving noise.

Two kind of questions:

- small noise: study the limit in distribution $\lim_{\sigma \to 0} x^\sigma$ and get then some informations on exit times from an interval or first passage times
- large time: fix $\sigma = 1$ and study the limits in distribution $\lim_{t \to \infty} v_t^1$ and $\lim_{t \to \infty} x_t^1$

I will just explain the kind of results for the first type of questions and focus on the second one. (If time I will try to explain the challenge for the extended Brox model).

Keep in mind the following main example

$$v_t^\sigma = -\int_{t_0}^{t} \frac{\mathrm{sign}(v_s^\sigma)|v_s^\sigma|^\beta}{s^\gamma} \mathrm{d}s + \sigma \ell_t \quad \text{and} \quad x_t^\sigma = \int_{t_0}^{t} v_s^\sigma \, \mathrm{d}s$$

with ℓ symmetric α-stable Lévy process, $\alpha \in (0, 2]$, $\beta \geq 1$, $\gamma \in \mathbb{R}$, initial conditions v_{t_0}, x_{t_0}. Note that $\alpha = 2$ corresponds to a Brownian driving noise.

Two kind of questions:

- small noise: study the limit in distribution $\lim_{\sigma \to 0} x^\sigma$ and get then some informations on exit times from an interval or first passage times
- large time: fix $\sigma = 1$ and study the limits in distribution $\lim_{t \to \infty} v_t^1$ and $\lim_{t \to \infty} x_t^1$

I will just explain the kind of results for the first type of questions and focus on the second one. (If time I will try to explain the challenge for the extended Brox model).

Lévy process

A real process L is a *Lévy process* if L is càdlàg,

$$\forall s \leq t, \quad L_t - L_s \perp\!\!\!\perp \sigma(L_u;\ u \leq s) \text{ and } L_t - L_s \sim L_{t-s}.$$

There exists (a Lévy triplet)

- a drift $d \in \mathbb{R}$,
- a diffusion coefficient $\sigma^2 > 0$
- a jump measure ν on $\mathbb{R} \setminus \{0\}$ such that $\int (|y|^2 \wedge 1)\nu(\mathrm{d}y) < \infty$

such that $\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi L_t}] = \mathrm{e}^{-t\psi(\xi)}$ with

$$\psi(\xi) := -\mathrm{i}d\xi + \frac{1}{2}\sigma^2\xi^2 + \int \left(1 - \mathrm{e}^{\mathrm{i}\xi y} + i\xi y \mathbf{1}_{\{|y| \leq 1\}}\right)\nu(dy)$$

L_t is essentially a sum of two independent processes : a continuous one (drifted Brownian motion $\sigma b_t - dt$) and a pure jump process ℓ_t.

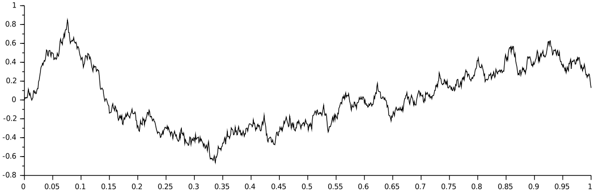

Figure: *Brownian motion*

L_t is essentially a sum of two independent processes : a continuous one (drifted Brownian motion $\sigma b_t - dt$) and a pure jump process ℓ_t.

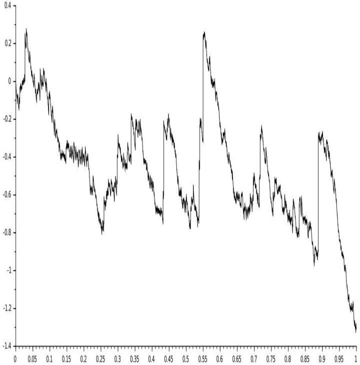

Figure: *Lévy process*

Stable pure jump process

ℓ *is a pure jump symmetric* α*-stable Lévy process*, $\alpha \in (0, 2)$, if

$$\mathrm{E}\left[e^{iu\ell(t)}\right] = e^{-tc_\alpha |u|^\alpha}, \quad \nu(dy) = \frac{1}{|y|^{\alpha+1}} \mathbf{1}_{\mathbb{R}\setminus\{0\}} dy.$$

It will be the driving noise all along the talk.

Remark : Brownian motion is a (continuous) 2-stable Lévy process.

Important feature of stable processes: self-similarity i.e.

$$\{\ell_t\} \quad \text{and} \quad \{c^{-1/\alpha} \ell_{ct}\} \quad \text{have the same law, for any} \quad c > 0.$$

Small noise and time-homogeneous drift

Slightly simplify our setting : $\gamma = 0$ (time-homogeneous), $t_0 = 0$ and initial conditions $v_0 = x_0 = 0$. The intention is to let $\sigma \to 0$ in our main equations

$$v_t^\sigma = -\int_0^t \mathrm{sign}(v_s^\sigma)|v_s^\sigma|^\beta \mathrm{d}s + \sigma \ell_t \quad \text{and} \quad x_t^\sigma = \int_0^t v_s^\sigma \mathrm{d}s$$

I will not discuss here the existence and uniqueness of the solution (even if not trivial). By self-similarity $\{\mathcal{L}_t^\sigma := \sigma \ell_{\sigma^{-\alpha} t}\}$ is also an α-stable process. Set

$$\mathcal{V}_t^\sigma := v_{\sigma^{-\alpha} t}^\sigma \quad \text{and} \quad \mathcal{X}_t^\sigma := x_{\sigma^{-\alpha} t}^\sigma$$

which satisfy respectively,

$$\mathcal{V}_t^\sigma = \mathcal{L}_t^\sigma - \frac{1}{\sigma^\alpha} \int_0^t \mathrm{sgn}(\mathcal{V}_s^\sigma)|\mathcal{V}_s^\sigma|^\beta \mathrm{d}s \quad \text{and} \quad \mathcal{X}_t^\sigma = \frac{1}{\sigma^\alpha} \int_0^t \mathcal{V}_s^\sigma \mathrm{d}s.$$

Denote $\theta := \frac{\alpha}{\alpha+\beta-1} > 0$ provided that $\alpha + \beta - 1 > 0$ and set (scaling)

$$L_t^\sigma := \frac{\mathcal{L}_{t\,\sigma^{\alpha\theta}}^\sigma}{\sigma^\theta} = \frac{\ell_{t\,\sigma^{-(\beta-1)\theta}}}{\sigma^{(\beta-1)\theta/\alpha}} \quad \text{and} \quad V_t^\sigma := \frac{\mathcal{V}_{t\,\sigma^{\alpha\theta}}^\sigma}{\sigma^\theta}.$$

L^σ is again a symmetric α-stable Lévy process and we have

$$V_t^\sigma = L_t^\sigma - \int_0^t \operatorname{sgn}(V_s^\sigma)|V_s^\sigma|^\beta \mathrm{d}s \;\text{ and }\; \mathcal{X}_t^\sigma = \sigma^{(2-\beta)\theta} \int_0^{t\sigma^{-\alpha\theta}} V_s^\sigma \mathrm{d}s.$$

(notice that the distribution of V^σ does not depend on σ).

Theorem (Gaussian asymptotic behaviour) (Eon and G., 2015)

As $\sigma \to 0$, the process $\{\sigma^{(\beta-1)\theta}\mathcal{X}_t^\sigma\} = \{\sigma^{(\beta-1)\theta}x_{\sigma^{-\alpha}t}^\sigma\}$ converges in distribution in $\mathrm{C}([0,\infty);\mathbb{R})$ to $\{\kappa_{\alpha,\beta}W_t\}$, provided $\beta + \frac{\alpha}{2} > 2$.
Here W is a real standard Brownian motion and $\kappa_{\alpha,\beta}$ is a positive constant having an integral representation.

Remarks:

- Interesting for physicians since we get a diffusive behaviour when starting with a dynamic involving jumps.
- Conditions $\alpha \in (0, 2)$ and $\beta + \frac{\alpha}{2} > 2$ force that $\beta > 1$ (and $0 < \theta < 1$): it is an over-damped drift;
- Hintze and Pavlyukevich (2014) studied the linear case $\beta = 1$: V^σ verifies a linear equation and $\mathcal{X}_t^\sigma$ is an integrated OU-process, it is explicit and this allows the study of the characteristic function.
- Kulik and Pavlyukevich (2019): after our work they obtained a non-Gaussian limit theorem (still stable), but taking a slightly different normalisation than ours.

Step 1 We prove existence and study the solution $g_{\alpha,\beta}$ of the Poisson equation $(\mathcal{A}_{\alpha,\beta}\, g)(x) = x$, where the infinitesimal generator of $V = V^\sigma$ is given by:

$$(\mathcal{A}_{\alpha,\beta}\, g)(x) = -\mathrm{sgn}(x)|x|^\beta g'(x) + \int_{\mathbb{R}} \left[g(x+y) - g(x) - yg'(x)\mathbf{1}_{|y|\le 1} \right] \nu(\mathrm{d}y).$$

Step 2 By the Itô-Lévy formula we get:

$$\sigma^{\theta(\beta+\frac{\alpha}{2}-2)} \mathcal{X}_t^\sigma = -\sigma^{\frac{\alpha\theta}{2}} M_{t\,\sigma^{-\alpha\theta}} + \sigma^{\frac{\alpha\theta}{2}} g_{\alpha,\beta}\left(V_{t\,\sigma^{-\alpha\theta}}\right),$$

with the square integrable càdlàg martingale

$$M_t := \int_0^t \int_{\mathbb{R}} [g_{\alpha,\beta}(z+V_s) - g_{\alpha,\beta}(V_s)] \tilde{N}(\mathrm{d}s, \mathrm{d}z)$$

($\tilde{N}$ being a compensated Poisson measure).

Step 3 We show that the remainder $\sigma^{\frac{\alpha\theta}{2}} g_{\alpha,\beta}\left(V_{t\,\sigma^{-\alpha\theta}}\right)$ converges toward 0 uniformly in probability (this is technical : use a Lyapunov function).

Step 4 We study the càdlàg martingale term M_t and its quadratic variation process. Invoking the ergodic theorem we show that

$$\lim_{\sigma \to 0} \langle \sigma^{\frac{\alpha\theta}{2}} M_{\bullet\, \sigma^{-\alpha\theta}} \rangle_t = \kappa_{\alpha,\beta} t.$$

We use the FCLT for martingales (Whitt's theorem) to show that $\sigma^{\frac{\alpha\theta}{2}} M_{t\, \sigma^{-\alpha\theta}}$ converges toward a BM with diffusion coefficient $\kappa_{\alpha,\beta}$.

About the ergodicity: need to impose $\beta > 1$ in order to prove that V is exponentially ergodic. There exists an invariant distribution $m_{\alpha,\beta}$ and assuming the integrability condition $\beta + \frac{\alpha}{2} > 2$, the ergodic theorem holds. It can be proved that

$$\kappa_{\alpha,\beta} := \iint_{\mathbb{R}^2} [g_{\alpha,\beta}(z+y) - g_{\alpha,\beta}(y)]^2 \nu(\mathrm{d}z) m_{\alpha,\beta}(\mathrm{d}y)$$

$$= -2 \int_{\mathbb{R}} x g_{\alpha,\beta}(x) m_{\alpha,\beta}(\mathrm{d}x) > 0.$$

Step 5 We conclude by Slutsky's theorem and the continuous mapping theorem.

Large time and time-inhomogeneous case

Let us return to our initial example (with $\sigma = 1$ and drop the superscript)

$$v_t = v_{t_0} - \int_{t_0}^{t} \frac{F(v_s)}{s^\gamma} \mathrm{d}s + \ell_t \quad \text{and} \quad x_t = x_{t_0} + \int_{t_0}^{t} v_s \mathrm{d}s$$

with ℓ symmetric α-stable Lévy process, $\alpha \in (0, 2]$, $F(v) = \mathrm{sign}(v)|v|^\beta$, $\beta \geq 1$, $\gamma \in \mathbb{R}$, initial conditions v_{t_0}, x_{t_0}. Our goal: try to find a rate r_ε such that $r_\varepsilon x_{t/\varepsilon}$ converges as a process, when $\varepsilon \to 0$ to a non-zero limit.

Similar questions were studied recently by Fournier and Tardif (2021) in the time-homogeneous case (inspired by physics).

Skip questions on existence, uniqueness and non-explosion of the solution: once again not really routine since we can't use classical tools for time-homogeneous sde's. For Brownian noise $\ell = b$ ($\alpha = 2$) the existence, uniqueness and behaviour in large time of the velocity process v was studied by Offret and G. in 2013.

Our system can be written

$$r_\varepsilon x_{t/\varepsilon} = r_\varepsilon x_0 + \int_{\varepsilon t_0}^{t} \frac{r_\varepsilon}{\varepsilon} v_{s/\varepsilon} \mathrm{d}s$$

and, setting $\tilde{r}_\varepsilon = r_\varepsilon/\varepsilon$,

$$\tilde{r}_\varepsilon v_{t/\varepsilon} = \tilde{r}_\varepsilon(v_0 - \ell_{t_0}) + \tilde{r}_\varepsilon \ell_{t/\varepsilon} - (\tilde{r}_\varepsilon)^{1-\beta} \varepsilon^{\gamma-1} \int_{\varepsilon t_0}^{t} \frac{F(\tilde{r}_\varepsilon v_{s/\varepsilon})}{s^\gamma} \mathrm{d}s$$

Hence the self-similarity of ℓ_t implies that $\tilde{r}_\varepsilon = \varepsilon^{1/\alpha}$ in order to get that $\tilde{r}_\varepsilon \ell_{\bullet/\varepsilon}$ converges in distribution . This forces $r_\varepsilon = \varepsilon^{1+1/\alpha}$.

Let me give some results which we obtained with Luirard. For both situations, Brownian noise ($\alpha = 2$) or pure jump (Lévy) stable noise there are three regimes. For simplicity I will state the results only for the stable case with $\alpha \in (1,2)$ (but true for any α betwen 0 and 2) and $\beta \in (1 - \alpha/2, \alpha)$.

Theorem 2 (large time behaviour) (Luirard and G., 2022)

Fix $\alpha \in (1,2)$ and assume $\beta \in (1-\alpha/2, \alpha)$. Set $\eta := \frac{\gamma}{\alpha+\beta-1}$. The limit of $\left(\varepsilon^{1/\alpha} v_{\bullet/\varepsilon}, \varepsilon^{1+1/\alpha} x_{\bullet/\varepsilon}\right)$, as $\varepsilon \to 0$ is as follows :

- **over-critical case,** $\eta > 1/\alpha$**:** kinetic limit toward $\left(\ell_\bullet, \int_0^\bullet \ell_s \mathrm{d}s\right)$;
- **critical case,** $\eta = 1/\alpha$**:** kinetic limit toward $\left(\mathcal{V}_\bullet, \int_0^\bullet \mathcal{V}_s \mathrm{d}s\right)$, where $\mathcal{V}_t = t^{1/\alpha} H_{\log t}$, H being the stationary solution of the homogeneous SDE $dH_t = \mathrm{d}\ell_t - H_t/\alpha\,\mathrm{d}t - F(H_t)\mathrm{d}t$ starting from its invariant distribution;
- **under-critical case,** $\eta < 1/\alpha$**:** only fdd limit of $\left(\varepsilon^{1/\alpha} v_{\bullet/\varepsilon}\right)$ toward the image under $u \mapsto t^\eta u$ of $(\widetilde{H}_t)$ the solution of the homogeneous SDE $d\widetilde{H}_t = \mathrm{d}\ell_t - F(\widetilde{H}_t)\mathrm{d}t$.

Remarks:

- The conditions on η can be also written as $\alpha\gamma > \alpha + \beta - 1$ (or $=$, or $<$).
- It can be show that there is no convergence as a process of $\left(\varepsilon^{1/\alpha} v_{\bullet/\varepsilon}\right)$ in the under-critical case.
- Taking a Brownian noise ($\alpha = 2$) and considering the linear case $\beta = 1$, if $\gamma \in (-1/2, 1)$ (so again under-critical case) we can show that $\left(\varepsilon^{\gamma+1/2} x_{\bullet/\varepsilon}\right)$ tends fdd to a Gaussian process whose covariance is explicit.
- Extensions for Lévy processes satisfying some technical conditions are also available
- At our knowledge there are no references on the subject for Lévy driving noise nor for friction force varying in time.

An important tool in the proof (and interesting on its own) is the moment estimates for the velocity process, solution of the SDE

$$v_t = v_{t_0} - \int_{t_0}^{t} \frac{F(v_s)}{s^\gamma}\,\mathrm{d}s + \ell_t, \quad \text{where} \quad F(v) = \mathrm{sign}(v)|v|^\beta.$$

Theorem 3 (moment estimates) (Luirard and G., 2022)

For any $\alpha \in (0,2]$, $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}$ and $\kappa \in [0,\alpha)$, there exists a constant $C = C_{\gamma,\kappa,\beta,t_0}$ such that, for all $t \geqslant t_0$, $\mathbb{E}\left[|v_t|^\kappa\right] \leqslant C\,t^{\frac{\kappa}{\alpha}}$.

Note that the above bounds are the best possible, taking $F = 0$.

- If $\alpha = 2$, getting these estimates is standard and based on classical Itô's formula applied with the function $g : v \mapsto |v|^\kappa$, with $\kappa = 2$ or $\kappa > 2$ and employing martingale tools and Jensen inequality.
- For $\alpha \in (0,1)$, the pure jump process is $\ell_t = \sum_{s \leq t} \Delta \ell_s$. The estimate is obtained by applying Itô-Lévy's formula with the function $g_n : v \mapsto \sqrt{v^2 + 1/n}$,

$$g_n(v_t) = g_n(v_0) - \int_{t_0}^t g_n'(v_s) F(v_s)/s^\gamma \mathrm{d}s + \sum_{s \leq t} (g_n(v_s) - g_n(v_{s-})),$$

(and noting that $\sum_{s \leq t} |\Delta \ell_s|$ is also a stable process).
- When $\alpha \in [1,2)$ and key idea is to slice the small and big jumps in a non-homogeneous way

$$\ell_t - \ell_{t_0} = \int_{t_0}^t \int_{|z| \leq \xi^{\frac{1}{\alpha}}} z \widetilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_{t_0}^t \int_{|z| > \xi^{\frac{1}{\alpha}}} z N(\mathrm{d}s, \mathrm{d}z) - \int_{t_0}^t \int_{|z| > \xi^{\frac{1}{\alpha}}} z \nu(\mathrm{d}z) \mathrm{d}s.$$

Itô-Lévy formula is applied with the function $g : v \mapsto (\zeta + v^2)^{\kappa/2}$, $\zeta > 0$, optimise in ζ and make a good choice of ξ.

The second main ingredient is a space-time scaling transformation. Pick a smooth diffeomorphism $\varphi : [0, t_1) \to [t_0, +\infty)$ and set $R_t = \int_0^t \frac{\mathrm{d}\ell_{\varphi(s)}}{\varphi'(s)^{\frac{1}{\alpha}}}$ and $v_t^{(\varphi)} = \frac{v_{\varphi(t)}}{\varphi'(t)^{\frac{1}{\alpha}}}$. Then by self-similarity R is also an α-stable process and

$$\mathrm{d}v_t^{(\varphi)} = \mathrm{d}R_t - \frac{\varphi'(s)^{1-\frac{1}{\alpha}}}{\varphi(t)^{\gamma}} F(v_t^{(\varphi)})\mathrm{d}t - \frac{1}{\alpha}\frac{\varphi''(t)}{\varphi'(t)} v_t^{(\varphi)}\mathrm{d}t.$$

- exponential change in critical case $\varphi : t \mapsto t_0 e^t$ gives a homogeneous SDE: $\mathrm{d}v_t^{(\varphi)} = \mathrm{d}R_t - (t_0 e^t)^{-(\alpha+\beta-1)(\eta-1/\alpha)} F\big(v_t^{(\varphi)}\big)\mathrm{d}t - v_t^{(\varphi)}/\alpha \, \mathrm{d}t$
- power change in sub-critical case $\varphi : t \mapsto \big(t_0^{1-\alpha q} + (1-\alpha\eta)t\big)^{\frac{1}{1-\alpha\eta}}$ gives a non-homogeneous SDE: $\mathrm{d}v_t^{(\varphi)} = \mathrm{d}R_t - F\big(v_t^{(\varphi)}\big)\mathrm{d}t - \eta\varphi(t)^{\alpha\eta-1} v_t^{(\varphi)}\mathrm{d}t$, but " almost" homogeneous SDE.

Lemma (Offret and G, 2013)

Let Z and H be regular strong Markov processes which are, respectively, weak solutions of the SDE with continuous coefficients,

$$dZ_t = \sigma(t, Z_t)d\ell_t + b(t, Z_t)dt \quad \text{and} \quad dH_t = \sigma_\infty(H_t)d\ell_t + b_\infty(H_t)dt$$

Assume that Z is "asymptotically time-homogeneous", that is uniformly on compact subsets of $\mathbb{R}$,

$$\lim_{t\to\infty} \sigma(t, z) = \sigma_\infty(z) \quad \text{and} \quad \lim_{t\to\infty} b(t, z) = b_\infty(z),$$

H converges in distribution to Π, and Z is bounded in probability.
Then Z converges also in distribution to Π.

Lemma (Gronwall's type lemma)

Fix $r \in [0, 1)$ and $t_0 \in \mathbb{R}$. Assume that g is a non-negative real-valued function, b is a positive function and a is a differentiable real-valued function. Moreover, suppose that the function bg^r is a continuous function.
Assume that

$$\forall t \geq t_0,\ g(t) \leq a(t) + \int_{t_0}^{t} b(s) g(s)^r \mathrm{d}s.$$

Then, setting $C_r := 2^{\frac{1}{1-r}}$,

$$\forall t \geq t_0,\ g(t) \leq C_r \left[a(t) + \left((1-r) \int_{t_0}^{t} b(s) \mathrm{d}s \right)^{\frac{1}{1-r}} \right].$$

Possible extensions :

- add an extra term in the SDE : $-\nabla U(x_t)\mathrm{d}t$, modelling an evolution into a potential U

$$v_t = v_{t_0} - \int_{t_0}^{t} \frac{F(v_s)}{s^\gamma}\mathrm{d}s - \int_{t_0}^{t} \nabla U(x_s)\mathrm{d}s + \ell_t \quad \text{and} \quad x_t = x_{t_0} + \int_{t_0}^{t} v_s \mathrm{d}s$$

  Luirard and Cavallazzi 2022 proved some results of large time behaviour for a quadratic potential U, that means a linear additional term (confining effect on the position)

- keep this quadratic potential and replace the driving noise by a Ornstein-Uhlenbeck process (based on Brownian or Lévy process): recent works on MCMC methods turn around similar models ...

- ...

Time varying random environment

Some questions from ocean modelling point to the following equation

$$v_t = v_{t_0} - \int_{t_0}^{t} \frac{\dot{W}(v_s)}{s^\gamma} \mathrm{d}s + b_t \quad \text{and} \quad x_t = x_{t_0} + \int_{t_0}^{t} v_s \mathrm{d}s,$$

where W is a Brownian motion indexed by $\mathbb{R}$, that is
$W(x) = W_x^+ \mathbb{1}_{x \geq 0} + W_{-x}^- \mathbb{1}_{x<0}$, with W^+, W^- two independent standard BM ("case $\beta = -1/2$")

When $\gamma = 0$ the velocity process is the Brox diffusion (1986). The large time behaviour of v was studied by Offret for $\gamma \geq 1/4$ (2014). What about x ?
Also open problem for ℓ instead b.
The equation is formal and a sense is given by using the infinitesimal generator. The approach should be based on martingale problem considerations (ongoing work)...

Thank you!