Reflecting Brownian motions and last passage percolation

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Based on joint work with
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A classical fact

$$\sup_{t\geq 0}(B^{(-\mu)}(t)) \stackrel{d}{=} e(2\mu)$$

where

- $B^{(-\mu)}$ is a Brownian motion with drift $-\mu$ where $\mu > 0$,
- $e(2\mu)$ is a random variable which has the exponential distribution with rate 2μ.

Theorem 1 (F., Warren)

$$\sup_{t \geq 0} \lambda_{\max}(H(t) - tD) \stackrel{d}{=} \max_{\pi \in \Pi_n^{\text{flat}}} \sum_{(i,j) \in \pi} e_{ij}$$

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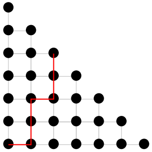
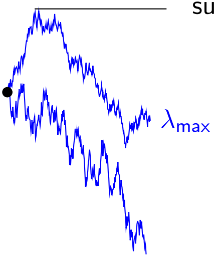
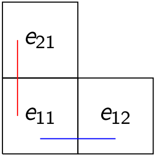
- $(H(t) : t \geq 0)$ is an $n \times n$ Hermitian Brownian motion, and D is an $n \times n$ diagonal matrix with entries $D_{jj} = \alpha_j > 0$.
- e_{ij} are an independent collection of exponential random variables indexed by $\mathbb{N}^2 \cap \{(i,j) : i + j \leq n+1\}$ with rates $\alpha_i + \alpha_{n+1-j}$.
- Π_n^{flat} is the set of up/right paths from $(1,1)$ to the line $\{(i,j) : i + j = n+1\}$.



The case $n = 2$



$$\sup_{t \geq 0} \lambda_{\max}(H(t) - tD)$$



$$\max(e_{11} + e_{21}, e_{11} + e_{12})$$

Eigenvalues of Hermitian Brownian motion

Let $(H(t) - tD : t \geq 0)$ be an $n \times n$ Hermitian Brownian motion with drift matrix D.

In the case $D = 0$ the eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$ satisfy

$$d\lambda_i(t) = \sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt + dB_i(t).$$

This can be interpreted as n independent Brownian motions conditioned not to collide using a Doob h-transform with the harmonic function

$$\Delta(x) = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Representation for largest eigenvalue of Hermitian Brownian motion

A result from Baryshnikov (fixed t and $D = cI$), O'Connell Yor (processes in t and $D = cI$) and Assiotis, O'Connell, Warren (process in t and general D) is that

$$\lambda_{\max}(H(t) - tD) \stackrel{d}{=} \sup_{0=t_0 \le t_1 \le \ldots \le t_n = t} \sum_{i=1}^{n} (B_i^{(-\alpha_{n-i+1})}(t_i) - B_i^{(-\alpha_{n-i+1})}(t_{i-1})).$$

- This can be interpreted as the highest particle in a system of Brownian motion reflecting off one another.
- Alternatively, interpret as last passage percolation in a white noise environment on $\{1, 2, \ldots, n\} \times [0, \infty)$.

Point-to-line last passage percolation

- Has important connections to
  - TASEP with a periodic initial conditons (sites of $\mathbb{Z}$ alternately occupied/empty).
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$$\frac{1}{c_2 n^{1/3}} \left(\max_{\pi \in \Pi_n^{\text{flat}}} \sum_{(i,j) \in \pi} e_{ij} - c_1 n \right) \overset{(n \to \infty)}{\to} F_1$$

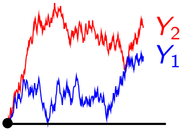
where F_1 is the Tracy-Widom GOE distribution.

A system of reflected Brownian motions

A system of reflected Brownian motions can be defined starting from zero as

$$Y_1(t) = \sup_{0 \le s \le t} (B_1^{(-\alpha_1)}(t) - B_1^{(-\alpha_1)}(s))$$

$$Y_j(t) = \sup_{0 \le s \le t} (B_j^{(-\alpha_j)}(t) - B_j^{(-\alpha_j)}(s) + Y_{j-1}(s)) \text{ for } j \ge 2.$$



The wall introduces an extra time variable to take a supremum over. After time reversal,

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Y_n(t) \overset{d}{=} \sup_{0 \le s \le t} \lambda_{\max}(H(s) - sD) &\overset{(t \to \infty)}{\to} \sup_{s \ge 0} \lambda_{\max}(H(s) - sD) \\
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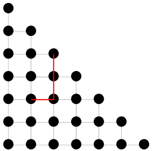
Q: What about the stationary distribution of all of the particles?

Theorem 2 (F., Warren)

The invariant measure of a system $(Y_1, Y_2, \ldots, Y_n)$ of reflected Brownian motions with a wall at the origin is given by a vector of last passage percolation times $(G(1,n), G(1,n-1), \ldots, G(1,1))$, where

$$G(k,l) = \max_{\pi \in \Pi_n^{\text{flat}}(k,l)} \sum_{(i,j) \in \pi} e_{ij}$$

where $\Pi_n^{\text{flat}}(k,l)$ is the set of all up/right paths from (k,l) to the line $\{(i,j) : i+j = n+1\}$.



Theorem 1 is a consequence of Theorem 2.

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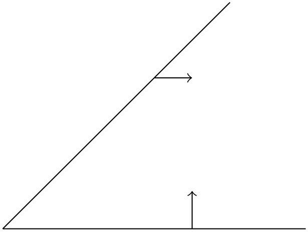

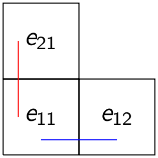

$$(e_{12}, \max(e_{11} + e_{21}, e_{11} + e_{12}))$$

We have found a process with an invariant measure $(G(1,n), G(1,n-1), \ldots, G(1,1))$.

Can we embed this within a higher-dimensional process with an invariant measure given by $(G(i,j) : i+j \leq n+1)$?

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Can we embed this within a higher-dimensional process with an invariant measure given by $(G(i,j) : i+j \leq n+1)$?

The field of last passage percolation times $(G(i,j) : i+j \leq n+1)$ is simpler in many ways than $G(1,1)$ since the local update rules are simple

$$G(i,j) = \max(G(i,j+1), G(i+1,j)) + e_{ij}.$$

The difficulty in LPP is having a convenient expression after marginalisation.

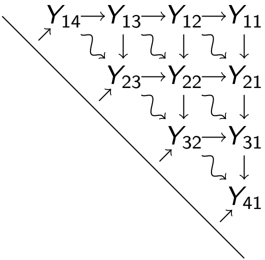
A system $\{Y_{ij} : i + j \leq n + 1\}$ satisfying

- For $j = 1, \ldots, n$

$$dY_{1j}(t) = dB_{1j}(t) - \alpha_{n-j+1}dt + dL^1_{1j}(t).$$

- For $i > 1$ and $i + j \leq n + 1$

$$\begin{aligned} dY_{ij}(t) = dB_{ij}(t) - \alpha_{n-j+1}1_{\{Y_{ij}<Y_{i-1,j+1}\}}dt \\ + \alpha_{i-1}1_{\{Y_{ij}>Y_{i-1,j+1}\}}dt + dL^1_{ij}(t) - dL^2_{ij}(t). \end{aligned}$$



We prove a positive temperature version of the statement that the invariant measure of $\{Y_{ij} : i + j \leq n + 1\}$ is given by the field of last passage percolation times $\{G(i,j) : i + j \leq n + 1\}$.

$$G(k,l) = \max_{\pi \in \Pi_n^{\text{flat}}(k,l)} \sum_{(i,j) \in \pi} e_{ij}$$

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Once run in stationarity the following dynamical reversibility property holds

$$(Y_{ij}^{(\alpha_1,\dots,\alpha_n)}(t))_{t\in\mathbb{R}, i+j\leq n+1} \stackrel{d}{=} (Y_{ji}^{(\alpha_n,\dots,\alpha_1)}(-t))_{t\in\mathbb{R}, i+j\leq n+1}.$$

A positive temperature setting

Let $(X_{ij}(t) : i + j \leq n + 1, t \geq 0)$ be a diffusion on $\mathbb{R}^{n(n+1)/2}$ with generator

$$\mathcal{L}f = \sum_{\{(i,j): i+j \leq n+1\}} \frac{1}{2} \frac{d^2 f}{dx_{ij}^2} + b_{ij}(\mathbf{x}) \frac{df}{dx_{ij}}$$

where $\mathbf{x} = \{x_{ij} : i + j \leq n + 1\}$ and

$$b_{ij}(\mathbf{x}) = -\alpha_{n-j+1} + \frac{(\alpha_{i-1} + \alpha_{n-j+1}) e^{x_{ij}}}{e^{x_{i-1,j+1}} + e^{x_{ij}}} 1_{\{i>1\}} + e^{-(x_{ij} - x_{i,j+1})} 1_{\{i+j<n+1\}} - e^{-(x_{i-1,j} - x_{ij})} 1_{\{i>1\}} + \frac{1}{2} e^{-x_{ij}} 1_{\{i+j=n+1\}}.$$

Let $\{W_{ij} : (i,j) \in \mathbb{N}^2, i+j \leq n+1\}$ be a family of independent inverse gamma random variables with shape parameters $\gamma_{ij} = \alpha_i + \alpha_{n-j+1}$.

Partition functions of the log gamma polymer introduced by Seppäläinen are given by for $k+l \leq n+1$,

$$\zeta_{kl} = \sum_{\pi \in \Pi_n^{\text{flat}}(k,l)} \prod_{(i,j)\in\pi} W_{ij},$$
$$\xi_{kl} = \log \zeta_{kl}.$$

Theorem 3 (F., Warren)
The unique invariant measure of the process $(X_{ij}(t) : i+j \leq n+1, t \geq 0)$ is given by $(\xi_{ij} : i+j \leq n+1)$.

Theorem 1 and 2 are consequences of Theorem 3.

Corollary

$$\int_{0=s_0<s_1<\ldots<s_n<\infty} e^{\sum_{i=1}^{n} B_i^{(-\alpha_i)}(s_i)-B_i^{(-\alpha_i)}(s_{i-1})} ds_1 \ldots ds_n$$

is equal in distribution to

$$2 \sum_{\pi \in \Pi_n^{\text{flat}}} \prod_{(i,j)\in\pi} W_{ij}$$

where W_{ij} is a collection of inverse gamma random variables indexed by $\mathbb{N}^2 \cap \{(i,j) : i+j \leq n+1\}$ with shape parameters $\alpha_i + \alpha_{n-j+1}$ and rate 1.

- The left hand side is the point-to-line partition function of the O'Connell Yor polymer.
- The right hand side is the point-to-line partition function of the log gamma polymer introduced by Seppäiläinen.

References:

- (with Jon Warren.) Point-to-line last passage percolation and the invariant measure of a system of reflected Brownian motions. *Probab. Theory Rel. Fields*. 178, 121–171, 2020.
- The invariant measure of PushASEP with a wall and point-to-line last passage percolation. *Electron. J. Probab.* 26, 1–26, 2021.

Thank you!