Multidimensional singular control and related Skorokhod problem: sufficient conditions for the characterization of optimal controls

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40 years of reflected Brownian motion and related topics
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(Our) singular control problem

Stochastic singular control problem: For each $x \in \mathbb{R}^d$, solve

$$\text{Minimize:} \quad J(x,v) = \mathbb{E}\Big[\int_0^\infty e^{-\rho t} h(X_t^{x;v})dt + \int_{[0,\infty)} e^{-\rho t} d|v|_t\Big]$$

$$\text{subject to} \quad dX_t^{x;v} = b(X_t^{x;v})\,dt + \sigma\,dW_t + e_1 dv_t, \quad X_{0-} = x,$$

where $\mathcal{V} := \{v : \Omega \times [0,\infty) \to \mathbb{R} : v$ is $\mathbb{F}$-adapted, càdlàg, bounded variation process$\}$ and $|v|$ denotes the total variation of the process v.

Example (linear-quadratic-singular control problem): b affine, $h(x) = (x - \hat{x})^2$

Singular control arises as the **limit problem**, as $n \to \infty$, of **bang-bang control problems**:

$$\text{Minimize:} \quad \mathbb{E}\Big[\int_0^\infty e^{-\rho t}(h(X_t^{x;v}) + |\alpha_t|)dt\Big], \quad \text{over processes s.t. } |\alpha_t| \le n,$$

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IDEA: replace $\int_0^t \alpha_s ds$ with v_t

A control $\bar{v} \in \mathcal{V}$ is **optimal if** $V(x) := J(x;\bar{v}) = \inf_{v\in\mathcal{V}} J(x;v)$.

QUESTION: CHARACTERIZATION OF $\bar{v}$?

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$$\text{(HJB)} \quad \max\{\rho V - LV - h, |V_{x_1}| - 1\} = 0, \quad \text{a.e. in } \mathbb{R}^d,$$

where $LV := bDV + \frac{\sigma^2}{2}\Delta V$ is the generator of the uncontrolled diffusion.

STRATEGY: solve (**HJB**) $\to$ find V $\to$ construct the optimal control v??

Define the **waiting region** $\mathcal{W} := \{|V_{x_1}| < 1\}$ and the **free-boundary** $\partial\mathcal{W}$

FACT: In many examples, the optimal control is exactly that process $\bar{v}$, with minimal total variation, which keeps the state process inside the domain $\overline{\mathcal{W}}$, by reflectig it in a direction prescribed by DV.

QUESTION (REFINED): characterization of $\bar{v}$ in terms of $\mathcal{W}$ and V_{x_1}?

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Our aim: find some sufficient conditions to characterize optimal policies, bypassing the problems related to the regularity of the **free boundary**.

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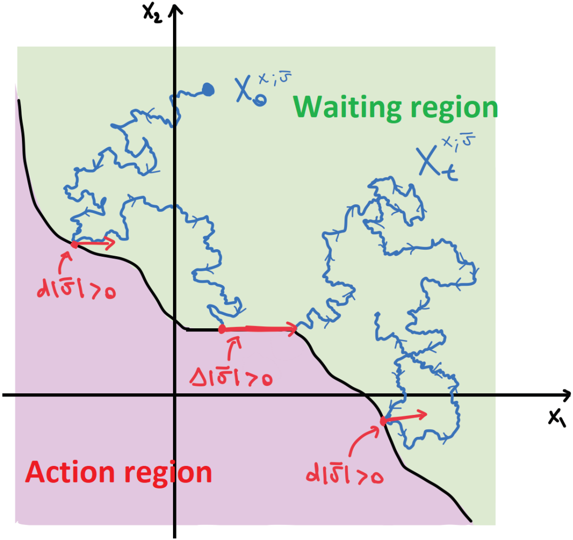
In a picture...



$\bar{v}$ = optimal control

$X^{x;\bar{v}}$ = optimally controlled state process

waiting region
$\mathcal{W} = \{|V_{x_1}| < 1\}$

Some extensions:

1. Convex-convex case: The dynamics is given by

$$\begin{cases} dX_t^1 = (a^1 + b_1^1 X_t^1)dt + \sigma dW_t^1 + dv_t \\ dX_t^i = b^i(X_t^1, X_t^i)dt + \sigma dW_t^i, \quad i = 2, ..., d, \end{cases}$$

for convex functions $b^i : \mathbb{R}^2 \to \mathbb{R}$ of class $C^{2,\mu}$ for any $\mu \in (0,1)$. Furthermore, we assume that the functions $b^i_{x_1}$, $b^i_{x_1 x_i}$, h_{x_i}, $h_{x_1 x_i} \geq 0$ for each $i = 2, ..., d$, and that Db^i is globally Lipschitz.

2. Geometric dynamics: consider a systems of type

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3. Degenerate dynamics: consider the model studied in [Federico et al. 2020, 2021]; that is,

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with $b_1^2 > 0$, $b_2^2 \leq 0$, $h_{x_1 x_2} \geq 0$. Idea: Exploit of the local Lipschitz continuity of the boundary proved in [Federico et al. 2021] to prove that V is C^2 inside $\mathcal{W}$.

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Idea of the proof

Idea of the proof:
- approximation of the free boundary in the spirit of [Kruk 2000]
- heavy use of control theoretical techniques

Key steps

1. The function V_{x_1} is the value of the 2-player zero-sum **game of optimal stopping** with evaluation functional ($\hat{h} := h_{x_1} + \beta DV$, $\bar{\rho} := \rho - B_1^1$, $\beta := (0, b_1^2)$)

$$G_x(\tau_1, \tau_2) := \mathbb{E}\left[\int_0^{\tau_1 \wedge \tau_2} e^{-\bar{\rho} t} \hat{h}(X_t^x) dt - e^{-\bar{\rho}\tau_1} \mathbb{1}_{\{\tau_1 \le \tau_2,\, \tau_1 < \infty\}} + e^{-\bar{\rho}\tau_2} \mathbb{1}_{\{\tau_2 < \tau_1\}} \right]$$

   by a variational formulation in the spirit of [Chiarolla&Haussmann 2000]

2. Monotonicity of V_{x_1} in the direction β, exployting the **structural assumption** on b and h (and comparison principles for SDEs);

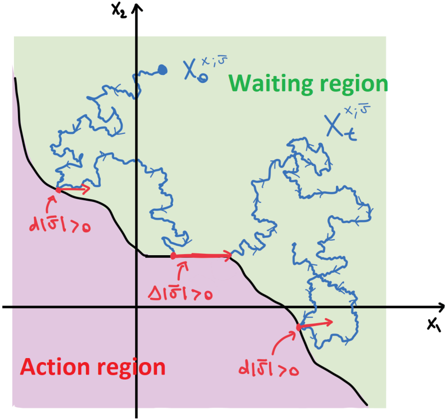
3. Construction of ε-optimal policies $\bar{v}^\varepsilon$ as solution to Skorokhod problem on domains $\mathcal{W}_\varepsilon := \{V_{x_1}^2 < 1 - \varepsilon\}$: crucial use of **Step 2** in order to invoke results from [Lions&Sznitman 1984];

4. Prove that $\bar{v}^\varepsilon \to \bar{v}$ as $\varepsilon \to 0$. Then, in the spirit of [Kruk 2000], the properties of $\bar{v}^\varepsilon$ and the optimality of $\bar{v}$ give the characterization result.

Singular control problem:

$$\text{Minimize:} \quad V(x) := \inf_{v \in \mathcal{V}} \mathbb{E}\Big[\int_0^\infty e^{-\rho t} h(X_t^{x;v}) dt + \int_{[0,\infty)} e^{-\rho t} d|v|_t \Big]$$

$$\text{subject to} \quad dX_t^{x;v} = b(X_t^{x;v})\, dt + \sigma\, dW_t + e_1 dv_t, \quad X_{0-} = x.$$



$\bar{v}$ = optimal control

$X^{x;\bar{v}}$ = optimally controlled state process

waiting region $\mathcal{W} = \{|V_{x_1}| < 1\}$

For more details: arXiv:2103.08487

THANK YOU FOR YOUR ATTENTION.

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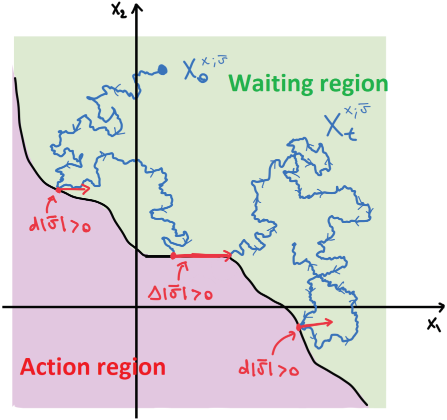

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