Infinite Atlas Model

Domains of Attraction, Extremality, and Fluctuations

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40 *Years of RBM and Related Topics, April 2023.*

Outline.

- Long time behavior of certain IPS.
- Model is an infinite dimensional RBM.
- Long time asymptotics of its f.d. analogue: well understood from Harrison and Williams.

Atlas model with $d + 1$ Brownian particles.

- $d + 1$ Brownian particles in $\mathbb{R}$. Lowest particle gets a drift of $+1$.
- Particle system with *topological interactions* [Carinci, A. De Masi, C. Giardina, and E. Presutti (2016)]: **Atlas model**.
- Stochastic portfolio theory. [Fernholz(2002), Fernholz and Karatzas(2009)].
- Connections with classical IPS (e.g. simple exclusion process) [Karatzas, Pal and Shklonikov (2016)]. ..
- ...also with nonlinear diffusions and McKean-Vlasov equations [Dembo, Shklonikov, Varadhan, Zeitouni (2016)]...
- .. and Aldous' *Up the River* stochastic control problem [Aldous 2002].

Finite Dimensional Atlas Model: Some Known Results

- Strong existence and pathwise uniqueness [Ichiba, Karatzas, Shklonikov[IKS] (2013)].
- Denote the ranked particle system as

$$Y_0(t) \leq Y_1(t) \leq \cdots Y_d(t)$$

and the gap process

$$Z_i(t) \doteq Y_i(t) - Y_{i-1}(t), \; 1 \leq i \leq d.$$

Then $Z = (Z_i)_{1 \leq i \leq d}$ is a Harrison-Reiman RBM in $\mathbb{R}_+^d$ with oblique reflections.

Finite Dimensional Atlas Model: Some Known Results

- The process Z is ergodic. The unique invariant probability measure is given as

$$\pi^{(d)} := \bigotimes_{i=1}^{d} \mathrm{Exp}(2(1 - i/(d+1))).$$

  [Harrison and Williams (1987a, 1987b), Pal and Pitman(2008).]

- Geometric ergodicity [B. and Lee (2007)], explicit dimension dependent rates [Banerjee and B. (2020), Banerjee and Brown (2021)] .

Infinite Atlas Model.

Now consider an infinite system of particles, i.e.

$$d = \infty.$$

Infinite Atlas Model: $d = \infty$.

- un-ranked particle processes: $U_i(t)$, ranked particle processes: $Y_i(t)$. Evolution:

$$dU_i(s) = \mathbf{1}(U_i(s) = Y_0(s))ds + dW_i(s), \; s \geq 0, i \in \mathbb{N}_0$$

  where $\{W_i\}_{i \in \mathbb{N}_0}$ are mutually independent Brownian motions

- Wellposedness. Suppose

$$\sum_{i=0}^{\infty} e^{-\alpha [U_i(0)]_+^2} < \infty, \; \forall \alpha > 0.$$

  Then strong existence and pathwise uniqueness holds for unranked system and the ranked process is well defined[Sarantsev(2017), IKS(2013)].

Infinite Atlas Model: $d = \infty$.

- SDE for ranked system:

$$dY_i(t) = \mathbf{1}(i=0)dt + dB_i(t) - \frac{1}{2}dL_{i+1}(t) + \frac{1}{2}dL_i(t),\ t \geq 0, i \in \mathbb{N}_0.$$

- $L_0(\cdot) \equiv 0$. $L_i(\cdot)$: collision local time of $(i-1)$-th and i-th particles. B_i are independent BM.

- Gap Process: $\mathbb{R}_+^\infty$-valued process $\mathbf{Z}(\cdot) = (Z_1(\cdot), Z_2(\cdot), \dots)$:

$$Z_i(\cdot) := Y_i(\cdot) - Y_{i-1}(\cdot),\ i \in \mathbb{N}.$$

- Goal: Study long-time behavior of $\mathbf{Z}(\cdot)$.

Infinite Atlas Model: The Gap process

- One invariant distribution: $\pi_0 \doteq \otimes_{i=1}^{\infty} \text{Exp}(2)$. [Pal and Pitman (2008).]
- Recall, d-dim. Atlas: unique inv. dist. is

$$\pi^{(d)} := \bigotimes_{i=1}^{d} \text{Exp}(2(1 - i/(d+1))).$$

- PP Conjecture(2008): π_0 is the unique invariant probability measure.
- Sarantsev and Tsai (2017) – False. These are all stationary:

$$\pi_a := \bigotimes_{i=1}^{\infty} \text{Exp}(2 + ia), \;\; a \geq 0.$$

Questions of Interest

- Local stability structure – Domains of attraction.
- Extremality/Ergodicity of $\{\pi_a\}$.
- π_a-Equilibrium fluctuations.

Domains of Attraction (DoA): Recent Results

- For a suitable $\nu \in \mathcal{P}(\mathbb{R}_+^\infty)$, $\hat{\nu}_t = \mathcal{L}(\mathbf{Z}(t))$, when $\mathcal{L}(\mathbf{Z}(0)) = \nu$.
- Let $\nu_t = \frac{1}{t}\int_0^t \hat{\nu}_s ds$.
- Definition.
  - ν is in the strong DoA of π_a if $\hat{\nu}_t \to \pi_a$ as $t \to \infty$.
  - ν is in the weak DoA of π_a if $\nu_t \to \pi_a$ as $t \to \infty$.
- Sarantsev (2017): If $\nu \overset{d}{\geq} \pi_0$ then $\nu \in \mathrm{sDoA}(\pi_0)$.

Domains of Attraction (DoA): Recent Results

- Dembo, Jara, Olla [DJO](2019): Let $\nu = \mathcal{L}(\mathbf{Z}(0))$, where $\mathbf{Z}(0) = (Z_j(0))_{j \in \mathbb{N}}$ a.s. satisfy:

$$\limsup_{m\to\infty} \frac{1}{m^{\beta}\theta(m)} \sum_{j=1}^{m} (\log Z_j(0))_{-} < \infty, \tag{1}$$

$$\liminf_{m\to\infty} \frac{1}{m^{\beta^2/(1+\beta)}\theta(m)} \sum_{j=1}^{m} Z_j(0) = \infty, \tag{2}$$

$$\limsup_{m\to\infty} \frac{1}{m^{\beta}\theta(m)} \sum_{j=1}^{m} Z_j(0) < \infty \tag{3}$$

Here: $\beta \in [1,2)$, $\inf_m\{\theta(m-1)/\theta(m)\} > 0$, $\{\theta(m)\}$ non-decreasing. Also, if $\beta = 1$, $\theta(m) \geq \log m$.

Then $\nu \in \mathrm{sDoA}(\pi_0)$.

Some Remarks

- $\nu \overset{d}{\geq} \pi_0$ says: particles not 'too closely packed'.
- DJO's third condition says: particles not too spread out.
    - E.g. $Z_j(0) \sim e^{j^2}$ not allowed by DJO but satisfies S's condition.
- Condition (1) of DJO requires all gaps to be strictly positive.
- Proofs: reversibility + Dirichlet forms and RE estimates.
- No results available for $a \neq 0$.

Some Challenges for $a \neq 0$.

- DJO approximate infinite Atlas by finite Atlas with $d+1$ particles:
  - Law of the first k gaps $\Rightarrow$ k-marginal of π_0, as $t \to \infty$ and $d \to \infty$ together, suitably.
  - For any T, $\exists\, d_T \in \mathbb{N}$ s.t. on $[0, T]$, d_T-dim. Atlas $\approx$ lowest $d_T + 1$ particles in infinite Atlas.

Some Challenges for $a \neq 0$.

- Crucial fact: *Unique* stationary measure of the d dim. Atlas, converges to π_0 as $d \to \infty$.
- So a finite Atlas approximation is 'tailor-made' to select π_0.
- For DoA of π_a – perhaps consider other f.d. approx. But coupling them with the infinite Atlas not clear.

DoA: Case $a \neq 0$.

- Coupling of two prob. meas. μ_1 and μ_2 on S_1 and S_2, is a pair (X_1, X_2) of $S_1 \times S_2$ valued r.v., s.t. $\mathcal{L}(X_i) = \mu_i$, $i = 1, 2$.

- What to Expect? Let W_i be iid $Exp(2)$. Then for $a, a' > 0$, with

$$V_{a,i} = \frac{2}{2 + ia} W_i, \;\; V_{a',i} = \frac{2}{2 + ia'} W_i, \;\; i \in \mathbb{N}$$

$V_a = (V_{a,i})_{i \in \mathbb{N}}$, $V_{a'} = (V_{a',i})_{i \in \mathbb{N}}$ is a coupling of π_a and $\pi_{a'}$.

- Clearly, $\pi_{a'} \notin \text{wDoA}(\pi_a)$. Also,

$$\sum_{i=1}^{d} |V_{a,i} - V_{a',i}| \sim O(\log d).$$

DoA: Case $a \neq 0$.

- Theorem [Banerjee and B. 2022] Fix $a > 0$. Suppose $\nu \in \mathcal{P}(\mathbb{R}_+^\infty)$ satisfies the following:

  There is a coupling $(\mathbf{U}, \mathbf{V}_a)$ of ν and π_a s.t., a.s.,

$$\limsup_{d\to\infty} \frac{\log\log d}{\log d} \sum_{i=1}^{d} |V_{a,i} - U_i| = 0, \text{ and } \limsup_{d\to\infty} \frac{U_d}{dV_{a,d}} < \infty.$$

  Then $\nu \in \mathrm{wDoA}(\pi_a)$.

  Conjecture. $(\log d)^{-1} \sum_{i=1}^{d} |V_{a,i} - U_i| \to 0$ is necessary.

Remarks.

- Corollary. Fix $a > 0$. Let

$$\nu \doteq \bigotimes_{d=1}^{\infty} \text{Exp}(2 + da + \lambda_d).$$

- Note. If $\lambda_d = da'$ for some $a' > 0$, clearly $\nu \notin \text{wDoA}(\pi_a)$.
- If $|\lambda_d| = o(d/\log\log d)$, then $\nu \in \text{wDoA}(\pi_a)$.

Main Result: Case $a = 0$.

- Theorem (Banerjee and B. (2022)) Suppose $\nu \in \mathcal{P}(\mathbb{R}_+^\infty)$ satisfies:

  there is a coupling $(\mathbf{U}, \mathbf{V})$ of ν and π_0 s.t., a.s.,

  $$\liminf_{d \to \infty} \frac{1}{\sqrt{d}(\log d)} \sum_{i=1}^{d} U_i \wedge V_i = \infty. \qquad (A)$$

  Then $\nu \in \mathrm{wDoA}(\pi_0)$.

- Note that if $\nu \overset{d}{\geq} \pi_0$ then condition holds.

- No upper bound constraints on gaps and gaps need not be positive.

Comparison with DJO.

DJO: Let $\mathbf{U} \sim \nu$. Suppose $U_j = \lambda_j \Theta_j$, Θ_j are iid, nonnegative, and

- $E\Theta_1 < \infty$, $E \log(\Theta_1)_- < \infty$.

Suppose one of the following hold:

(i) for some $c \in [1, \infty)$, $\lambda_j \in [c^{-1}, c]$ for all $j \in \mathbb{N}$.

(ii) $\Theta_1 \sim Exp(1)$ and one of the following holds:

(a) $\lambda_d \uparrow \infty$ and $\limsup_{d \to \infty} \frac{1}{d^\beta} \sum_{i=1}^d \lambda_i < \infty$ for some $\beta < 2$.

(b) $\lambda_d \downarrow 0$ and $\frac{1}{\sqrt{d} \log d} \sum_{i=1}^d \lambda_i \to \infty$, as $d \to \infty$.

Then $\nu \in sDoA(\pi_0)$.

Corollary (Banerjee and B.).

- Corollary. Let $\mathbf{U} \sim \nu$. Suppose $U_j = \lambda_j \Theta_j$, $j \in \mathbb{N}$, where $\{\Theta_j\}$ are iid and non-negative satisfying:
  $\cancel{E\Theta_1 < \infty}$, $\cancel{E \log(\Theta_1)^- < \infty}$ $\quad \mathbb{P}(\Theta_1 > 0) > 0$,

  and suppose one of the following holds:

  (i) ~~for some $c \in [1, \infty)$, $\lambda_j \in [c^{-1}, c]$ for all $j \in \mathbb{N}$.~~
  $\liminf_{j \to \infty} \lambda_j > 0$,

  (ii) $\cancel{\Theta_1 \sim Exp(1)}$ and one of the following holds:

  (a) ~~$\lambda_d \uparrow \infty$ and $\limsup_{d \to \infty} \frac{1}{d^\beta} \sum_{i=1}^d \lambda_i < \infty$ for some $\beta < 2$.~~ [See (i)]

  (b) $\cancel{\lambda_d \downarrow 0}$
  and $\frac{1}{\sqrt{d} \log d} \sum_{i=1}^d \lambda_i \to \infty$, as $d \to \infty$

  Then $\nu \in \cancel{\mathrm{sDoA}(\pi_0)}$ $\quad \mathrm{wDoA}(\pi_0)$.

Some Proof Ingredients.

- Synchronous coupling of infinite ordered particle dynamics starting from two initial conditions.
- Central ingredient: Estimates on the decay rate of the L^1 distance between the gaps.
- Analysis of excursions of the difference of the coupled processes.
- Each excursion contracts the L^1 distance by a fixed deterministic amount.
- Key: appropriately control the number and the lengths of such excursions.
- Other ingredients: Monotonicity properties of the gap process, quantifying the influence of far away coordinates on the first few gaps.
- Reduction to coupled systems with starting configurations differing only in finitely many coordinates.

Extremality of π_a.

- For $a \neq a'$, π_a and $\pi_{a'}$ are mutually singular (Kakutani's theorem).

- Extremality/Ergodicity of π_a?

- Definition An invariant prob. meas. ν is extremal if, whenever for some $\varepsilon \in (0,1)$ and invariant prob. ν_1, ν_2 we have $\nu = \varepsilon\nu_1 + (1-\varepsilon)\nu_2$, then $\nu_1 = \nu_2 = \nu$.

- Definition An invariant prob. meas. is ergodic if every (T_t)-invariant $\psi \in L^2(\nu)$ is constant ν-a.s.

- ν is extremal iff ν is ergodic.

Extremality: Previous works.

- Extremality in IPS on countably infinite graphs: Liggett(1976), Andjel(1982), Sethuraman(2001), Balázas, Rassoul-Agha, Seppäläinen, Sethuraman (2007).
- Interactions are Poissonian – effective generator methods.
- Interactions in Infinite Atlas are very 'singular' (local times); generator methods are less tractable.
- Also, state space is not countable; non-smooth boundary; oblique reflections.

Extremality: Main Result.

- Consider the general $\boldsymbol{g}$-Atlas model: drift vector $\boldsymbol{g} \in \mathcal{D}$, where

$$\mathcal{D} \doteq \left\{\boldsymbol{g} = (g_0, g_1, \dots) \in \mathbb{R}^\infty : \sum_{i=0}^{\infty} g_i^2 < \infty\right\}.$$

- Sarantsev(2017): All of the following are stationary distributions:

$$\pi_a^{\boldsymbol{g}} \doteq \otimes_{n=1}^{\infty} \mathrm{Exp}(n(2\bar{g}_n + a)), \quad a > -2 \inf_{n \in \mathbb{N}} \bar{g}_n,$$

  where $\bar{g}_n \doteq \frac{1}{n}(g_0 + \cdots + g_{n-1})$.

- $\boldsymbol{g} = (1, 0, \cdots)$ is the Atlas model.

- Harris(1965): $\boldsymbol{g}^0 = \boldsymbol{0}$. Extremality of $\{\pi_a^{\boldsymbol{0}} = \otimes_{n=1}^{\infty} \mathrm{Exp}(na)\}$ studied in Ruzmaikina-Aizenman(2005).

Extremality: Main Result.

- Theorem (Banerjee, B. (2022b)). For each $\boldsymbol{g} \in \mathcal{D}$ and $a > -2\inf_{n\in\mathbb{N}} \bar{g}_n$, $\pi_a^{\boldsymbol{g}}$ is extremal for the $\boldsymbol{g}$-Atlas model.

- If $\boldsymbol{g} \in \mathcal{D}_1$, where

$$\mathcal{D}_1 \doteq \{\boldsymbol{g}\in\mathcal{D} : \exists N_1 < N_2 < \ldots \to \infty \text{ s.t. } \bar{g}_k > \bar{g}_{N_j},\ k=1,\ldots,N_j-1, \forall j\geq 1\},$$

  $\pi_a^{\boldsymbol{g}}$ is also an extremal invariant distribution for

$$a = -2\inf_{n\in\mathbb{N}} \bar{g}_n.$$

- In particular for the infinite Atlas model π_a is extremal for all $a \geq 0$.

Proof Idea.

- Goal: (T_t)-invariant $\psi \in L^2(\pi_a^{\boldsymbol{g}})$ is constant $\pi_a^{\boldsymbol{g}}$-a.e.
- Consider first: $\boldsymbol{g} = \boldsymbol{g}^1$ and $a = 0$, $\pi_a^{\boldsymbol{g}} = \otimes_{i=1}^{\infty}\text{Exp}(2)$:
  - So the coordinate sequence $\{Z_i\}_{i=1}^{\infty}$ on $\mathbb{R}_+^{\infty}$ is iid under $\pi_a^{\boldsymbol{g}}$.
  - Hewitt-Savage: suffices to show that ψ is $\pi_a^{\boldsymbol{g}}$-a.e. invariant under all finite permutations of the coordinates... enough to consider transpositions of the i-th and $(i+1)$-th coordinates.

Proof Idea.

- Key: construction of a mirror coupling of the first $i+1$ BM, and synchronous coupling of the remaining BM of a pair of nearby initial configurations.

- Estimates on the probability of a successful coupling.

- General $\pi_a^{\mathbf{g}}$: More involved as $\{Z_i\}_{i=1}^{\infty}$ is not iid. Exploit scaling properties of Exponential distributions.

Open Question.

- Are these the only extremal invariant probability distributions?
- When $\boldsymbol{g} = \mathbf{0}$, Ruzmaikina-Aizenman(2005): Yes [under some integrability].
- Case of a general $\boldsymbol{g}$ (even $\boldsymbol{g}^1$) is open.
- Full characterization: simple exclusion process Liggett(1976), the zero range process Andjel(1982).
- Some features of these models: particle density for distinct extremal measures differ by a linear factor; monotonicity of ‘synchronously coupled’ dynamics; efficient generator methods.

Open Question.

- Some challenges for the Atlas Model:
  - singular interactions(local time)
  - point process of particles for $\pi_a^{\boldsymbol{g}}$ has an exponentially growing intensity when $a > 0$, a nonlinear dependence on a
  - the inhomogeneity of the topological interactions: local stability in a region of the particle cloud is affected both by the density of particles in the neighborhood and their relative ranks in the full system.

Answer to a Simpler Question.

- Are these the only extremal product form invariant probability distributions?
- Theorem (Banerjee and B. (2022)). Yes. (under an integrability condition)
- In fact these are the only product form invariant distributions (extremal or not).

Equilibrium Fluctuations.

- Dembo-Tsai(2017): Consider the π_0-stationary infinite Atlas model:

$$0 = Y_0(0) \leq Y_1(0) \leq Y_2(0) \cdots$$

- gaps $Y_{i+1}(0) - Y_i(0)$ iid distributed as Exp(2).

- Define for $x \geq 0$,
$\mathcal{X}^N(t,x) \doteq N^{-1/4}\left(2Y_{\lfloor N^{1/2}x/2\rfloor}(Nt) - \lfloor N^{1/2}x/2\rfloor\right).$

- Convergence of $\mathcal{X}^N(0,\cdot)$ to BM $\sqrt{2}B(\cdot)$ follows from Donsker's theorem.

Equilibrium Fluctuations.

- Theorem (Dembo-Tsai(2017)). $\mathcal{X}^N \Rightarrow \mathcal{X}$, in $C(\mathbb{R}_+^2 : \mathbb{R})$ where $\mathcal{X}$ solves the additive stochastic heat equation:

$$\left(\partial_t - \frac{1}{2}\partial_{xx}\right)\mathcal{X}(t,x) = \sqrt{2}\dot{\mathcal{W}},\ \ t,x > 0$$

with initial condition $\mathcal{X}(0,x) = \sqrt{2}B(x)$ and a Neumann boundary condition.

- $\{B(x) : x \geq 0\}$ is a Brownian motion and $\dot{\mathcal{W}}(t,x)$ is white noise on $\mathbb{R}_+^2$, independent of B.

- Question: What about the π_a-stationary infinite Atlas model?

Equilibrium Fluctuations: Ongoing (Banerjee, B. and Rudzis)

- Consider the Harris model, i.e. $\boldsymbol{g} = \mathbf{0}$, and for $a > 0$, the stationary measure $\pi_a^0 = \otimes_{n=1}^{\infty} \text{Exp}(na)$.

- Suppose $\{Y_i(0)\} \sim \text{PP}(ae^{ax}dx)$.

- Then $\{Y_{i+1}(0) - Y_i(0)\}_{i \in \mathbb{N}} \sim \pi_a^0$.

- Theorem (Liggett (1978)):
  For all t, $\{Y_i(t) + at/2\} \sim \text{PP}(ae^{ax}dx)$.

Equilibrium Fluctuations: Ongoing (Banerjee, B. and Rudzis)

- Let $Y_i^N(t) \doteq Y_i(t) + at/2 - \log N$.

- Let for $t \geq 0$ and $z > 0$,

$$\mathcal{X}^N(t,z) \doteq N^{a/2}\left[Y^N_{\lfloor N^a z \rfloor}(t) - \frac{1}{a}\log z\right].$$

- Then $\mathcal{X}^N \Rightarrow \mathcal{X}$ given as $\mathcal{X}(t,z) = \frac{1}{z}\mathcal{Y}(t,z)$, where $\mathcal{Y}$ solves:

$$\left(\partial_t - \frac{a^2x^2}{2}\partial_{xx}\right)\mathcal{Y}(t,x) = ax\dot{\mathcal{W}},\ \ t,x > 0$$

with initial condition $\mathcal{Y}(0,x) = B(x)$.