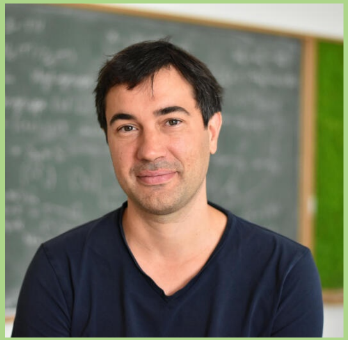
On the stationary distribution of RBM in a wedge

(arXiv 2021)

with Andrew Elvey Price, Sandro Franceschi, Charlotte Hardouin, and Kilian Raschel






Mireille Bousquet-Mélou
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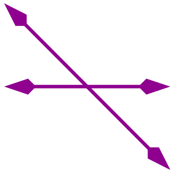
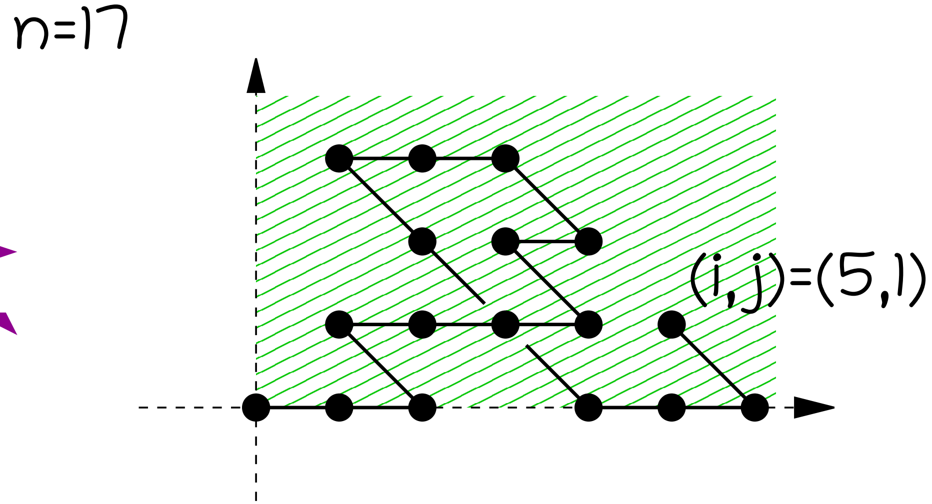
Enumerative combinatorics

How many trajectories of length n end at (i,j)?





Discrete

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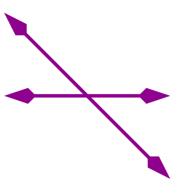

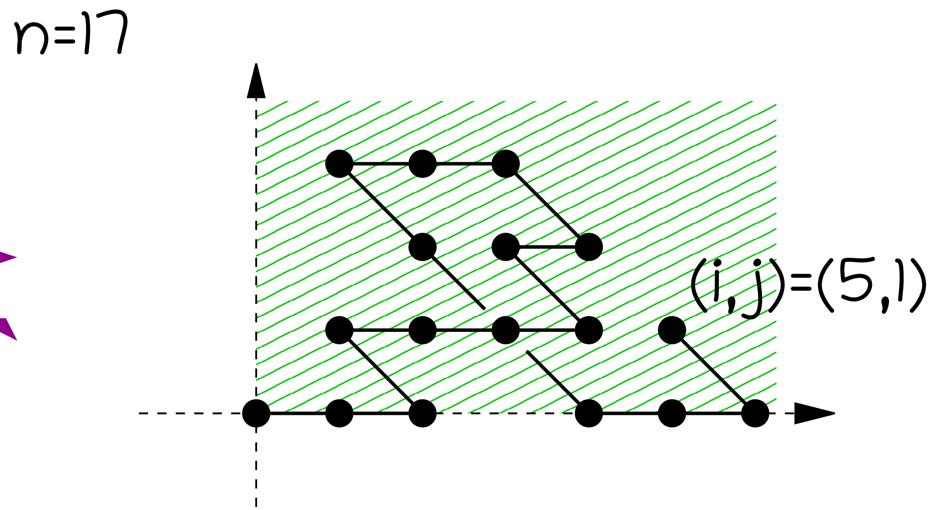

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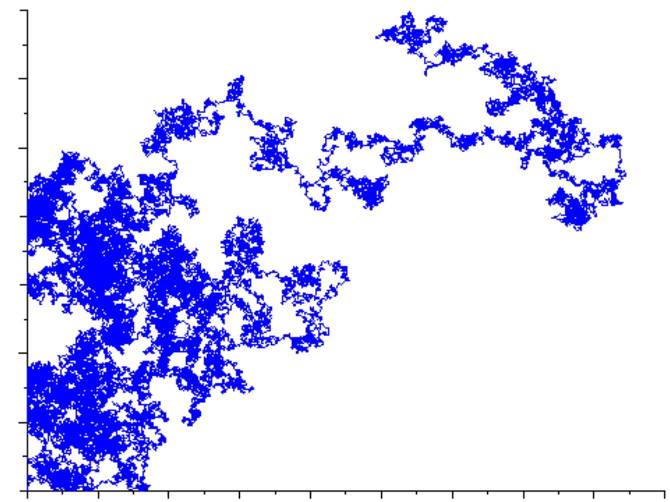
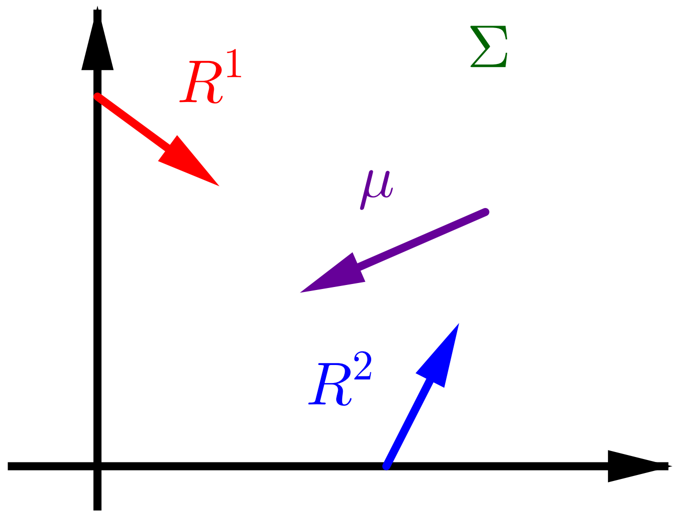


Continuous

I. Main results

Reflected Brownian motion in a quadrant



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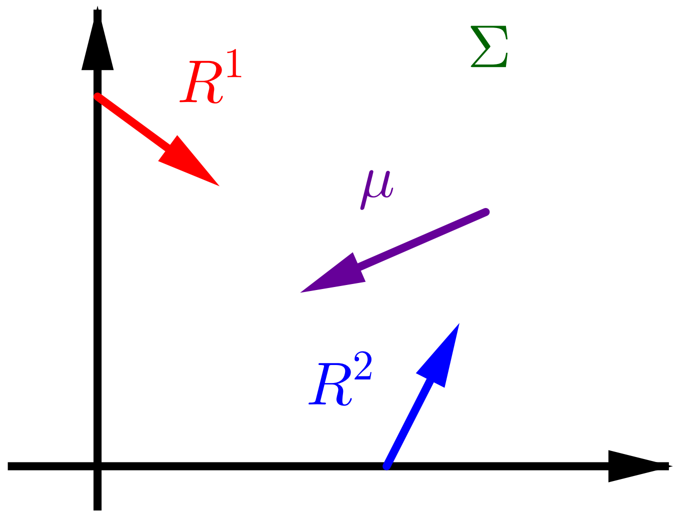
$$r_{11} > 0, \quad r_{22} > 0, \quad \det R > 0$$



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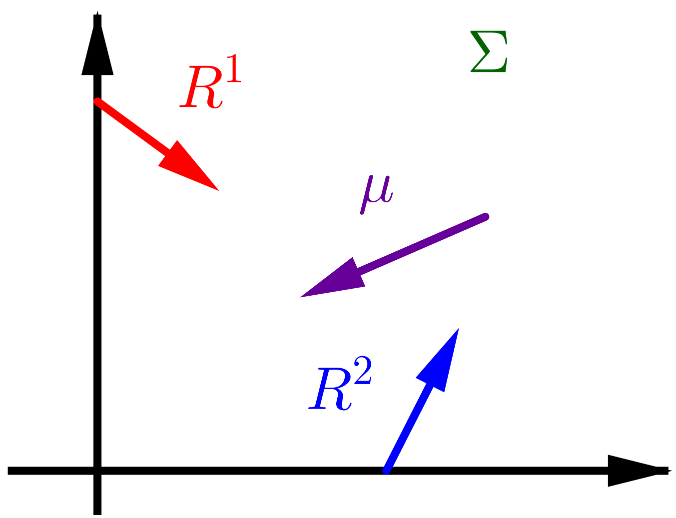

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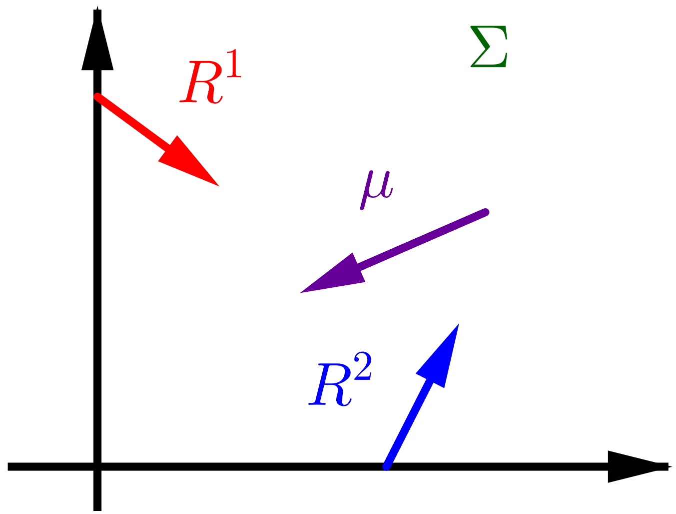

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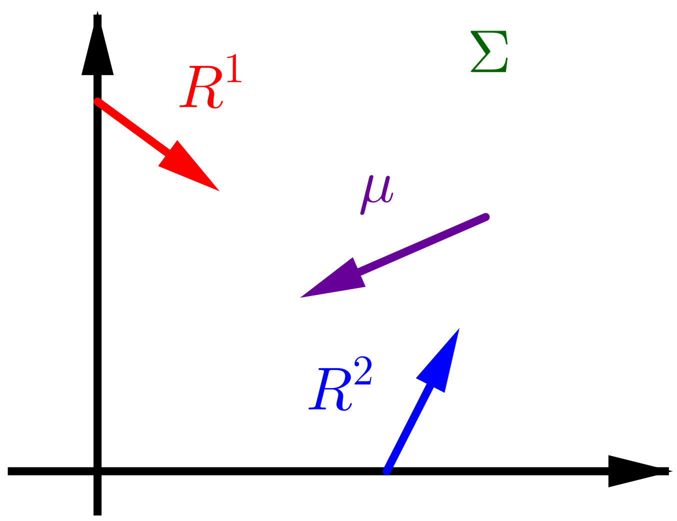
- negative drift $\quad \mu_1 < 0, \quad \mu_2 < 0$
- existence (and uniqueness) of stationary distribution

$$r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0$$



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[Varadhan & Williams 85, Hobson & Rogers 93]

Stationary distribution: the Laplace transform

- The stationary distribution has **density** $p_0(u,v)$, with **Laplace transform**:

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$$\boxed{-\gamma(x,y)\varphi(x,y) = \gamma_1(x,y)\varphi_1(y) + \gamma_2(x,y)\varphi_2(x)}$$

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$$r_{21}\varphi_1(y) = -(\mu_2 + \sigma_{22}\,y/2)\,\varphi(0,y) - r_{22}\frac{\mu_2 r_{11} - \mu_1 r_{21}}{r_{12}r_{21} - r_{11}r_{22}}$$

(same for $\varphi_2(x)$).

Stationary distribution: “simple” Laplace transform?

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Special cases:
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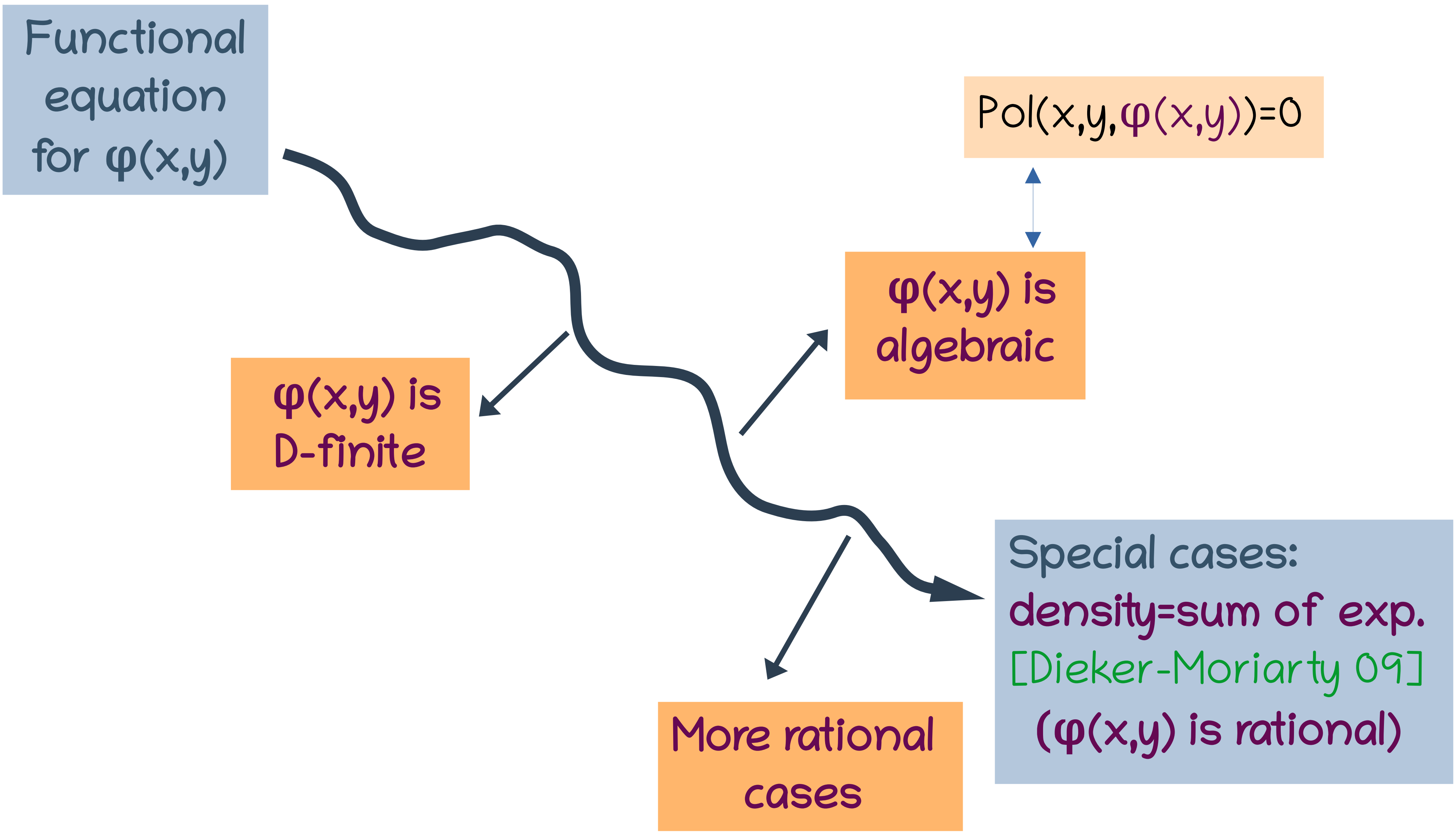

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Functional equation for $\varphi(x,y)$

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$\varphi(x,y)$ is algebraic

$\varphi(x,y)$ is D-finite

The moments $M_{m,n}$ are given by linear recurrence relations with polynomial coeffs.

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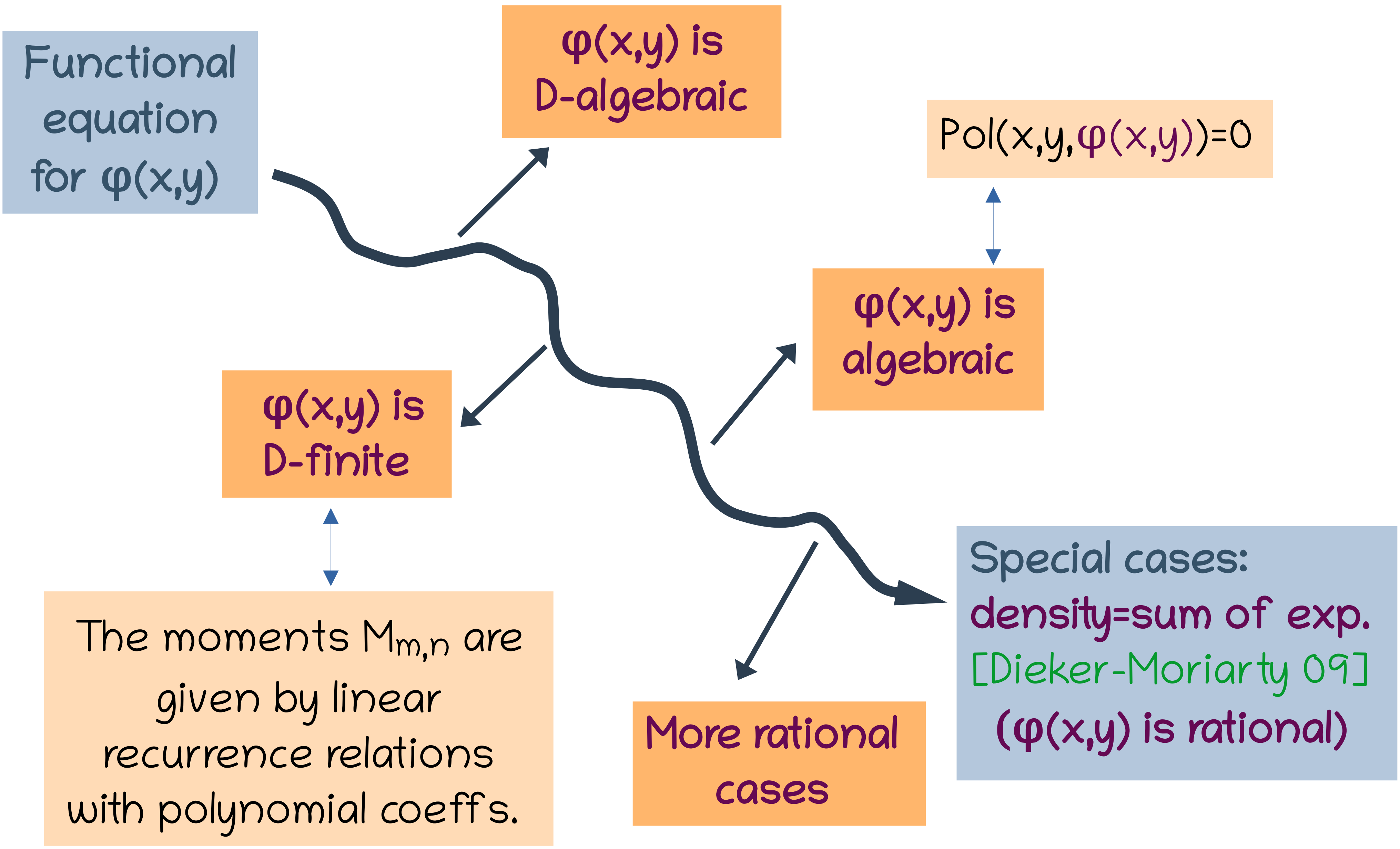

A hierarchy of functions

- Rational

$$\psi(x) = \frac{1-x}{1-x-x^2}$$

- Algebraic

$$1 - \psi(x) + x\psi(x)^2 = 0$$

- D-finite

$$x(1-16x)\psi''(x) + (1-32x)\psi'(x) - 4\psi(x) = 0$$

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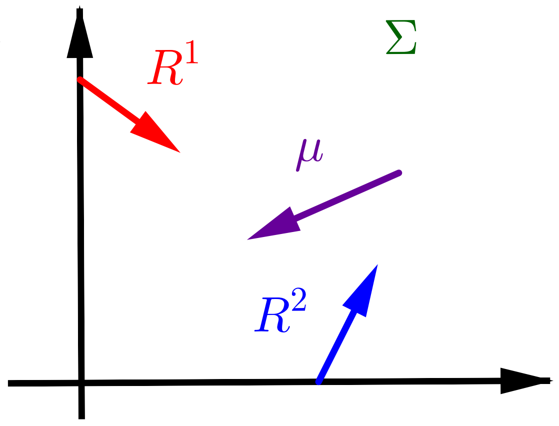
Multi-variate functions: one DE per variable

Main results (0)

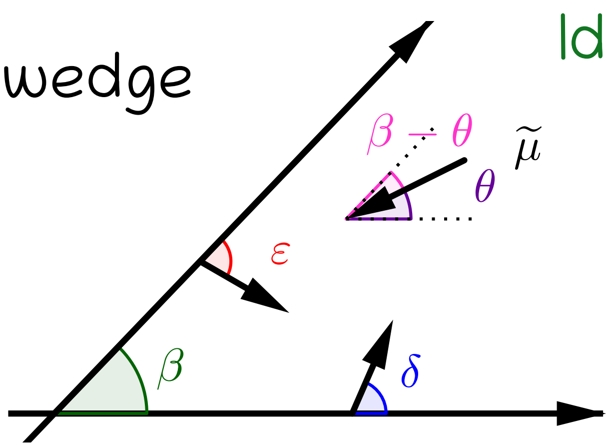
Two models in one

quadrant



Normalize the cone

β-wedge



Normalize the covariance matrix

4 angles

Main results (0)

A classical parameter:

$$\alpha = \frac{\delta + \varepsilon - \pi}{\beta}.$$

The process is a semi-martingale iff $\alpha < 1$ [Williams].

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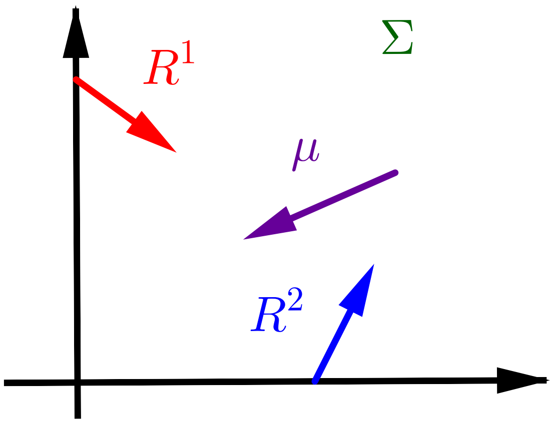

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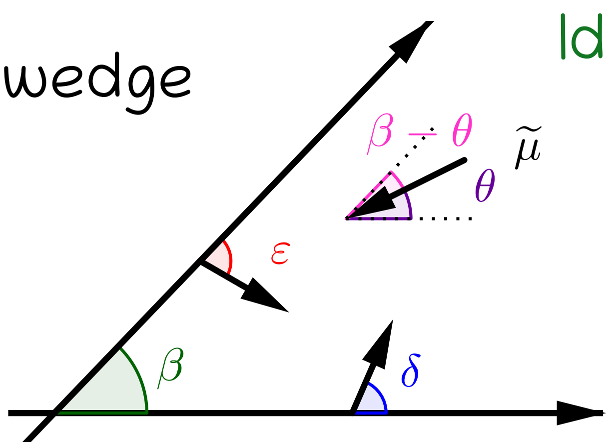

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A refinement (involves θ):

$$\alpha_1 = \frac{2\varepsilon + \theta - \beta - \pi}{\beta} \quad \text{and} \quad \alpha_2 = \frac{2\delta - \theta - \pi}{\beta},$$

(exchanged by symmetry). Note that $\alpha_1 + \alpha_2 = 2\alpha - 1$.

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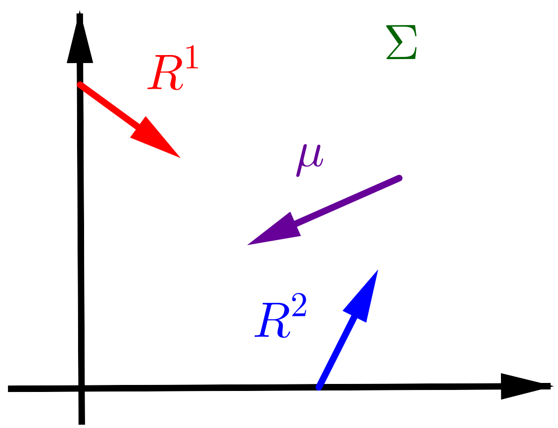

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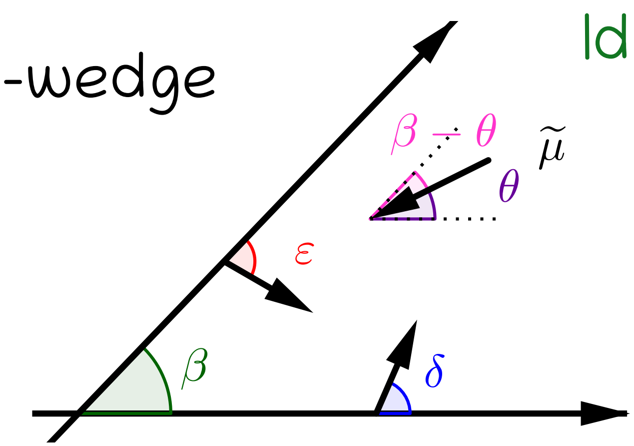

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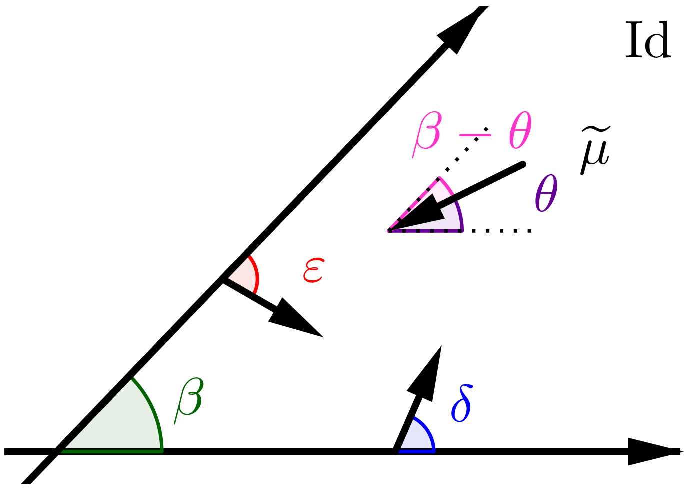

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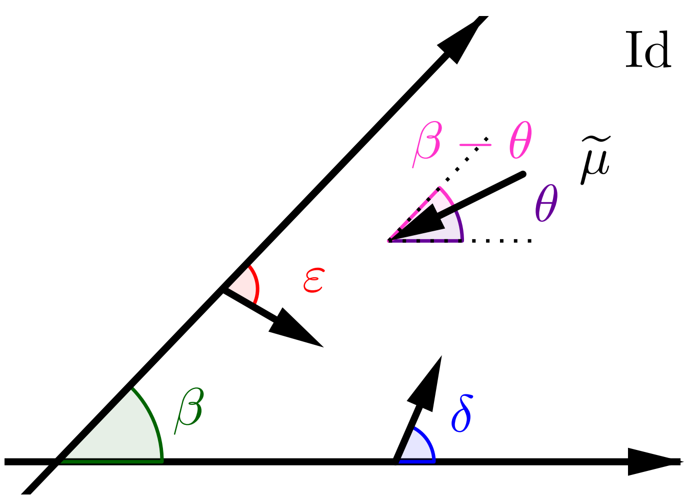

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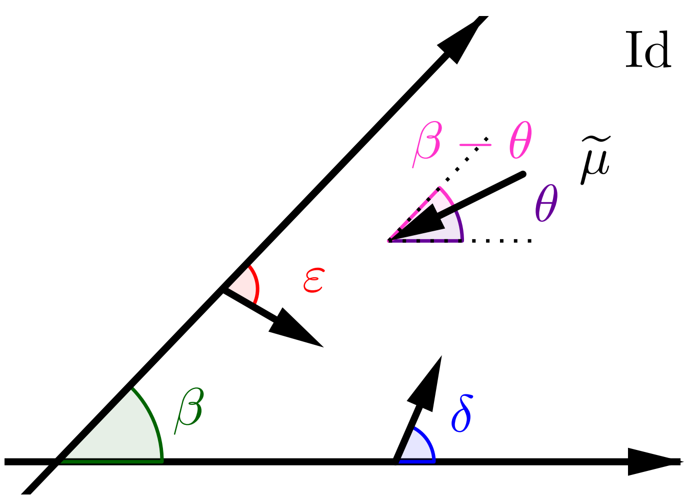

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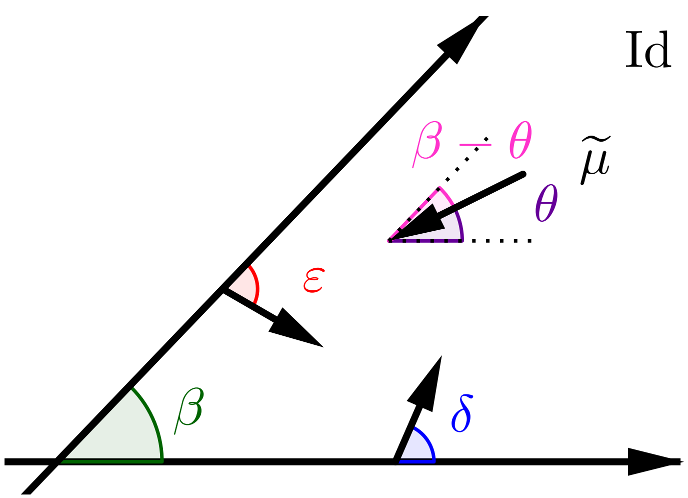

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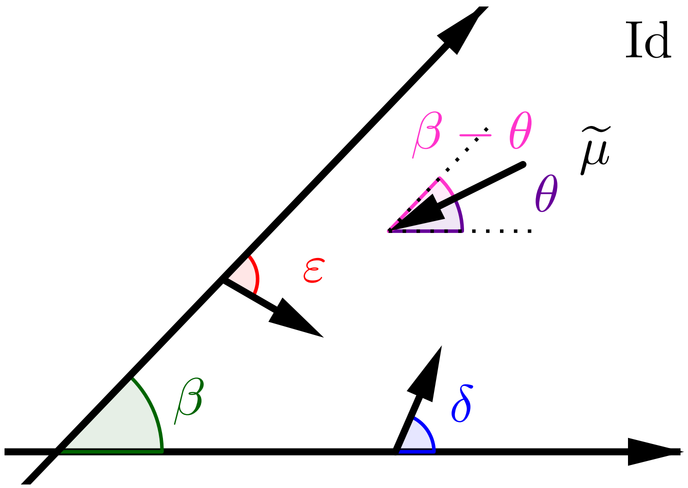

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Thm. As soon as Condition $\mathcal{C}$ holds, φ(x,y) has an elementary expression in terms of (possibly irrational) powers.

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More precisely, $\varphi_1(y)$ (or its square...) is a rational function in

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where a and b are explicit and $T_c(z)$ is a (D-finite) generalization of the Chebychev polynomial:

$$T_c(z) = \frac{1}{2}\left(\left(z+\sqrt{z^2-1}\right)^c + \left(z-\sqrt{z^2-1}\right)^c\right).$$

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$$(\mathcal{C}) \quad \alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}, \quad \text{or} \quad \{\alpha_1, \alpha_2\} \subset \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$$

More precisely, $\varphi_1(y)$ (or its square...) is a rational function in

$$y \quad \text{and} \quad T_{\pi/\beta}(ay+b),$$

trigonometry

where a and b are explicit and $T_c(z)$ is a (D-finite) generalization of the Chebychev polynomial:

$$T_c(z) = \frac{1}{2}\left(\left(z+\sqrt{z^2-1}\right)^c + \left(z-\sqrt{z^2-1}\right)^c\right).$$

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Corollary: $\varphi_1(y)$, $\varphi_2(x)$ and φ(x,y) are **D-algebraic** [closure properties].

II. Examples - Connections with previous work

Rational cases: $\delta + \varepsilon = \pi + m\beta$ ($\alpha \in \mathbb{Z}$) [simple]

Thm. Under this assumption,

$$\varphi_1(y) = \frac{1}{P(y)},$$

where $P(y)$ is a explicit polynomial.

Equivalently, the corresponding density $p_1(v)$ is a sum of terms

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Example. If $\delta + \varepsilon + \beta = \pi$ $(\alpha = -1)$ and $\theta - 2\delta = 2\beta + \pi$, then

$$\varphi_1(y) = \frac{\kappa}{(a-y)^2},$$

with density proportional to ve^{-av} (Erlang distribution).

A D-finite case: $\delta + \varepsilon + \beta = 2\pi$ [simple]

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$$\varphi_1(y) = \kappa \frac{T_{\pi/\beta}(ay+b) - A}{(B-y)(C-y)},$$

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- The function $\varphi_1(y)$ is **D-finite**, and **algebraic** iff $\pi/\beta \in \mathbb{Q}$.
- The linear differential equation satisfied by $T_c(z)$ yields an explicit 4^{th} order *recurrence relation* for the moments.

A "double" algebraic case: $\alpha_1 = \alpha_2 = 0$ [double]

Thm. This corresponds to

$$\theta = 2\delta - \pi, \qquad \beta - \theta = 2\varepsilon - \pi.$$

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- Explicit 2D density (in the β-wedge): in polar coordinates (r, a),

$$q_0(r\cos a, r\sin a) = \kappa \frac{\cos(\frac{\theta - a}{2})}{\sqrt{r}} \exp\left(-c\, r \cos^2\left(\frac{\theta - a}{2}\right)\right).$$

cf. [Harrison 78] in a special case ($\beta = \delta = \pi/2,\ \varepsilon = 3\pi/4,\ \mu_2 = 0$).

III. The proof: algebraic skeleton

(all analysis hidden)

A functional equation for $\varphi_1(y)$

The functional equation for $\varphi(x,y)$:

$$-\gamma(x, y)\varphi(x, y) = \gamma_1(x, y)\varphi_1(y) + \gamma_2(x, y)\varphi_2(x),$$

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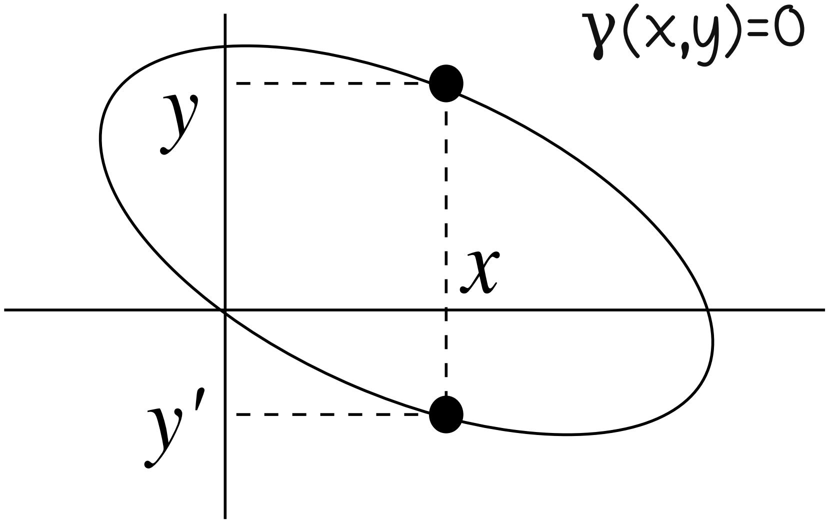
where the **kernel** $\gamma(x,y)$ is quadratic:

$$\gamma(x,y) = \frac{1}{2}(\sigma_{11}x^2 + 2\sigma_{12}xy + \sigma_{22}y^2) + \mu_1 x + \mu_2 y$$

and $\gamma_1(x,y)$, $\gamma_2(x,y)$ are linear polynomials in x,y.



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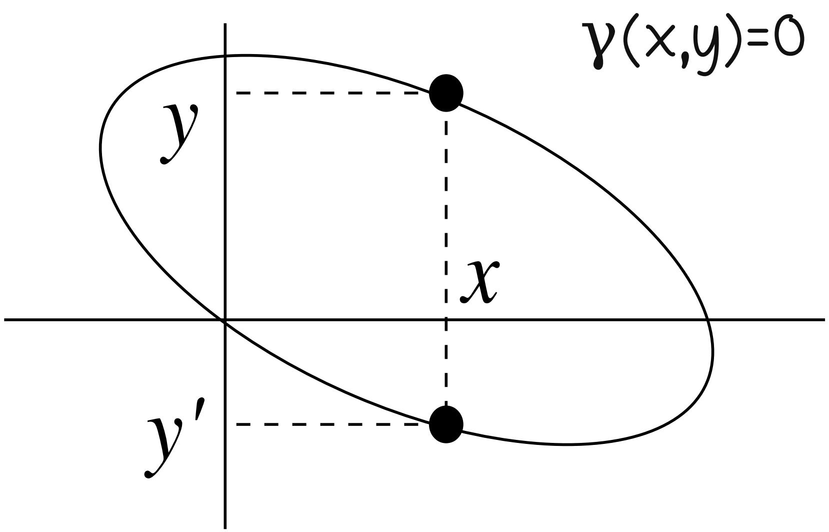

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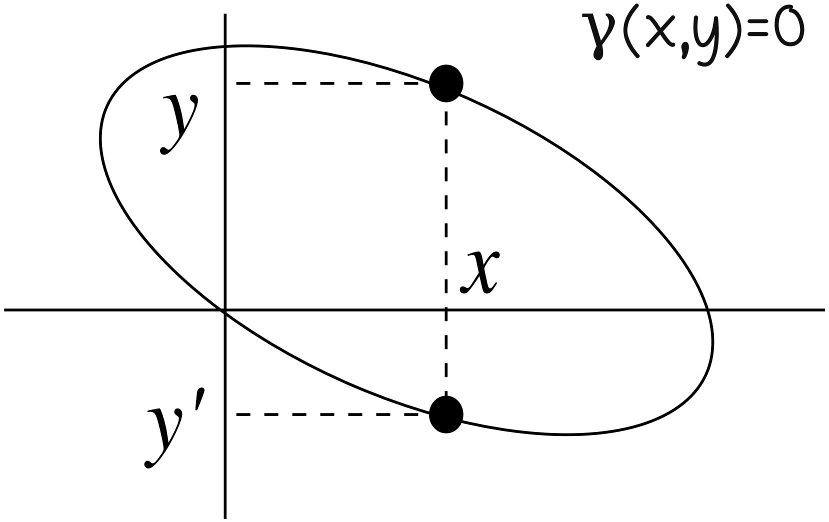

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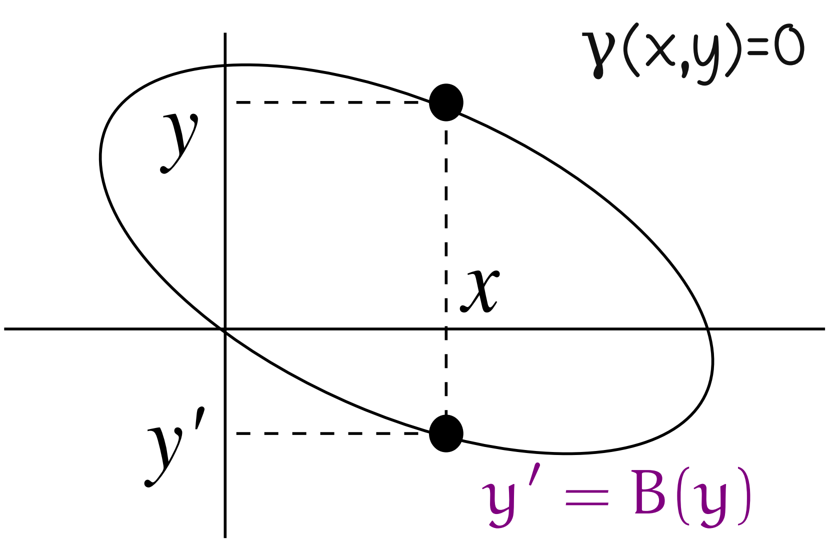
- This reads

$$\boxed{\varphi_1(B(y)) = A(y)\varphi_1(y),}$$

for **algebraic** functions A(y) and B(y).



A rational parametrisation of the curve γ(x,y)=0

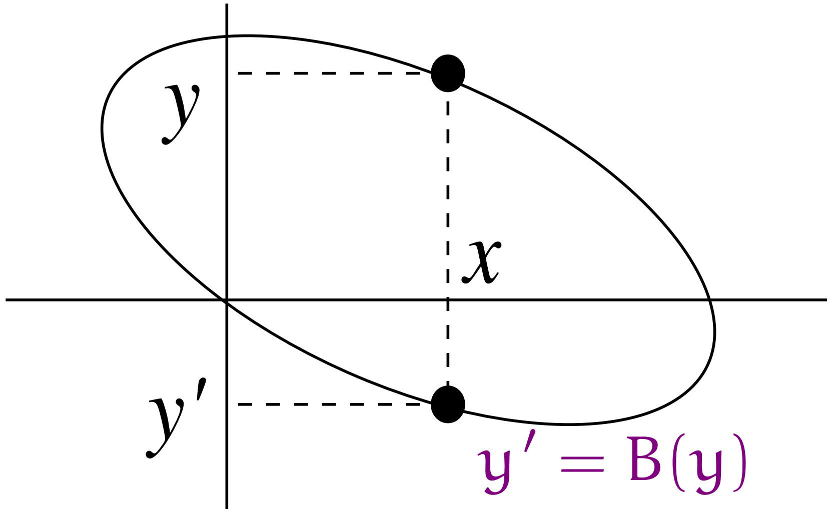
The curve

$$\gamma(x, y) = \frac{1}{2}(\sigma_{11}x^2 + 2\sigma_{12}xy + \sigma_{22}y^2) + \mu_1 x + \mu_2 y = 0$$

can be parametrized by

$$X(s) = a_1 + b_1 \left(s + \frac{1}{s}\right), \quad Y(s) = a_2 + b_2 \left(\frac{s}{e^{i\beta}} + \frac{e^{i\beta}}{s}\right),$$

with explicit constants.



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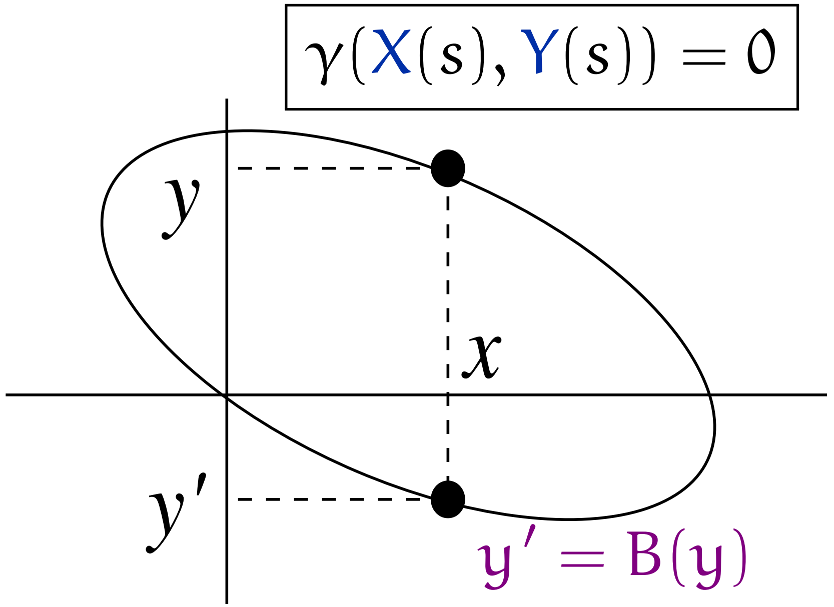

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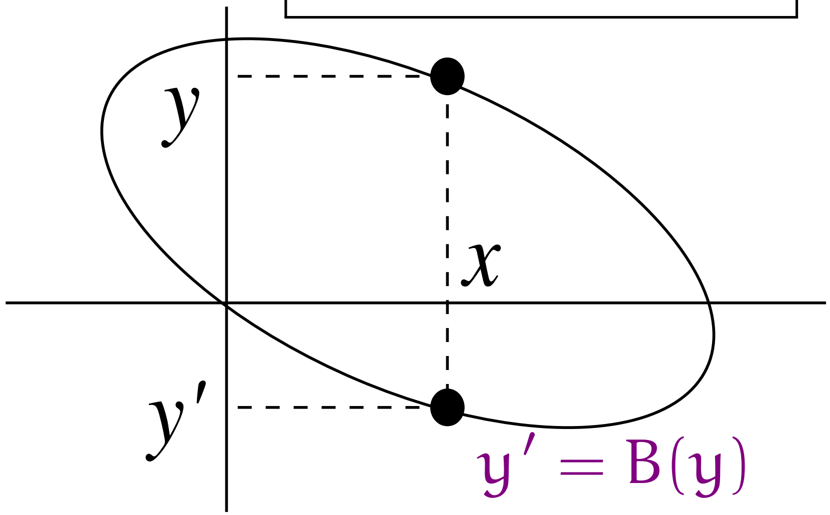
- If x=X(s), the two roots of γ(x,y) are y=Y(s) and y'=Y(1/s)= Y(sq), with q= $e^{2i\beta}$.

$$\gamma(X(s), Y(s)) = 0$$



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- If x=X(s), the two roots of γ(x,y) are y=Y(s) and y'=Y(1/s)= Y(sq), with q= $e^{2i\beta}$.

- The functional equation

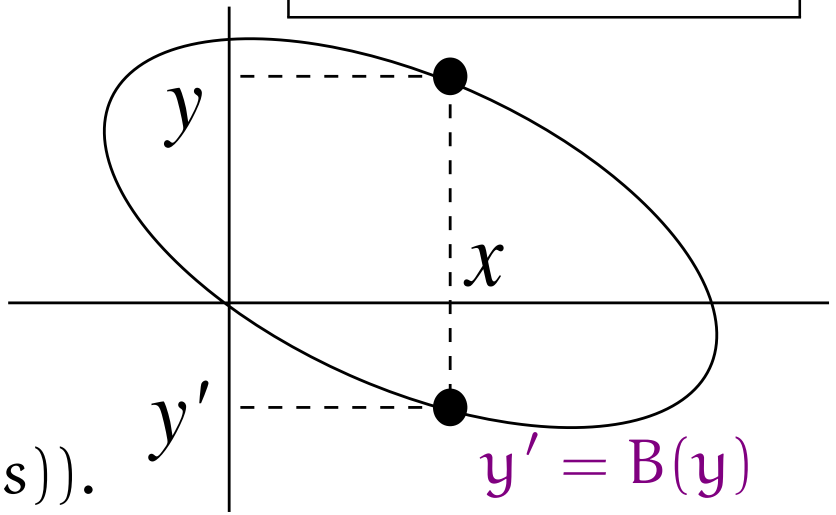
$$\varphi_1(B(y)) = A(y)\varphi_1(y),$$

becomes

$$\widetilde{\varphi}_1(sq) = \widetilde{A}(s)\widetilde{\varphi}_1(s),$$

where $\tilde{A}(s)$ is rational and $\widetilde{\varphi}_1(s) := \varphi_1(Y(s))$.

$$\gamma(X(s), Y(s)) = 0$$



Condition $\mathcal{C}$ ⇒ explicit solution (and D-algebraicity)

The new functional equation, with $\widetilde{\varphi}_1(s) := \varphi_1(Y(s))$ and $q = e^{2i\beta}$:

$$\boxed{\widetilde{\varphi}_1(sq) = \widetilde{A}(s)\widetilde{\varphi}_1(s).} \tag{1}$$

The rational function $\tilde{A}(s)$:

$$\widetilde{A}(s) = \frac{(s-s_1)(s_2 s-1)}{(s-s_2)(s_1 s-1)}, \quad \text{with } s_1 = -e^{i\beta(1-\alpha_1)},\ s_2 = -e^{i\beta\alpha_2}.$$

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⇒ explicit trigonometric solutions.

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D-algebraicity ⇒ Condition $\mathcal{C}$ [Galois theory]

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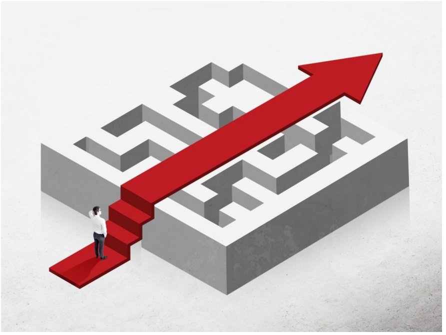

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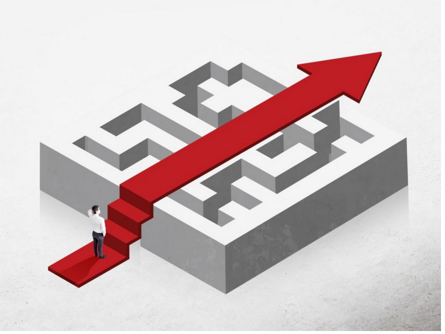


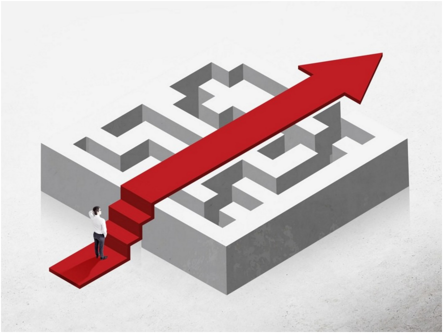
The Galois theory of q-difference equations gives a necessary condition on the function $\tilde{A}(s)$ for $\widetilde{\varphi}_1(s)$ (and $\varphi_1(y)$) to be D-algebraic.

D-algebraicity ⇒ Condition $\mathcal{C}$ [Galois theory]

The new functional equation, with $\widetilde{\varphi}_1(s) := \varphi_1(Y(s))$ and q= $e^{2i\beta}$:

$$\boxed{\widetilde{\varphi}_1(sq) = \widetilde{A}(s)\widetilde{\varphi}_1(s)}$$

for a rational function $\tilde{A}(s)$.



The Galois theory of q-difference equations gives a necessary condition on the function $\tilde{A}(s)$ for $\widetilde{\varphi}_1(s)$ (and $\varphi_1(y)$) to be D-algebraic.

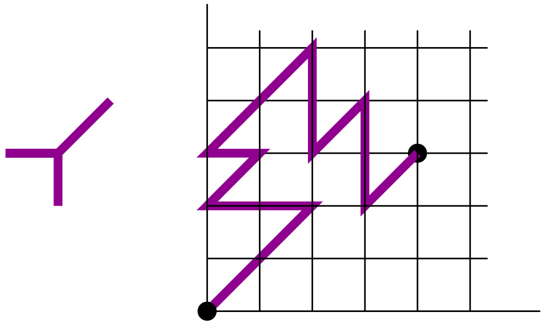
With our function $\tilde{A}(s)$... this boils down to Condition $\mathcal{C}$.

$$(\mathcal{C}) \quad \alpha \in \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}, \quad \text{or} \quad \{\alpha_1, \alpha_2\} \subset \mathbb{Z} + \frac{\pi}{\beta}\mathbb{Z}$$

Final comments

⊛ **Inspiration:** enumeration of **discrete lattice walks** in the quadrant:

$$(xy - t(x + y + x^2y^2))\, Q(t; x, y) = xy - txQ(t; x, 0) - tyQ(t; 0, y).$$



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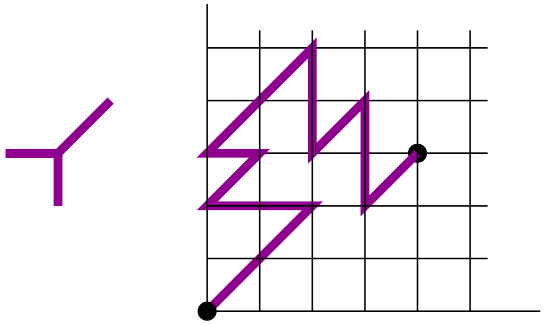

- **Two new tools** for the RBM:
  - Tutte's "invariant" theory (in disguise): explicit solutions
  - Galois theory of difference equations: necessary conditions for D-algebraicity

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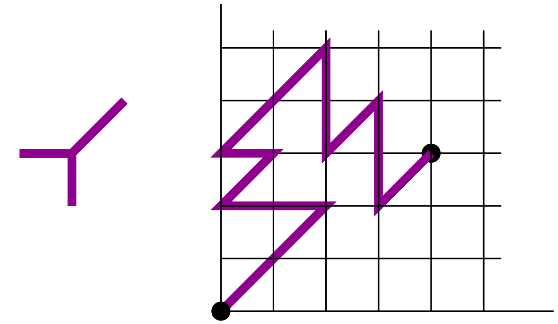

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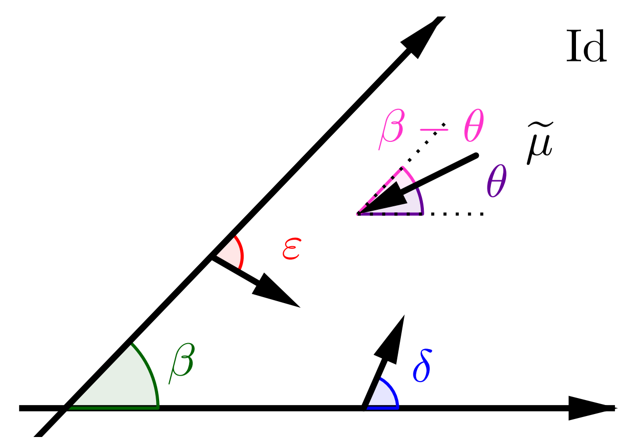

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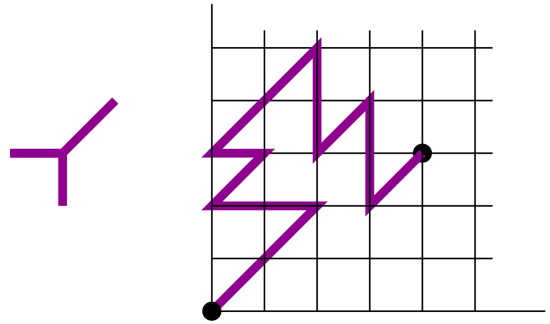
Thanks for your attention



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